A Review of Tests for Exponentiality

Andrey P. Rogozhnikov, Boris Yu. Lemesenko
Novosibirsk State Technical University, Novosibirsk, Russia

Abstract – A wide selection of tests for exponentiality is considered. Distributions of test statistics under true null hypothesis are studied and power of tests is estimated by means of methods of statistical simulation. A comparative analysis of power of tests with respect to competing alternatives with different shapes of hazard rate function is conducted. The conclusions are made on preference of one test or another under presence of specific competing alternatives.

Index terms – test, exponential distribution, power of test.

I. INTRODUCTION

A rather wide selection of tests for exponentiality is considered in some papers, e.g., [3, 4], and their power with respect to important competing hypotheses was studied by means of methods of statistical simulation. The results obtained made it possible to single out promising tests to apply in cases of having competing hypotheses. Of course, a set of goodness-of-fit tests could be applied but it appears from experience [1, 2] that the most powerful tests lie among the ones purposefully designed to test a hypothesis that sample follows one specific distribution.

A rather wide selection of tests for exponentiality is considered in some papers, e.g., [3, 4], and their power with respect to important competing hypotheses was studied by means of methods of statistical simulation. The results obtained made it possible to single out promising tests to apply in cases of having competing hypotheses with specific shape of hazard rate function and against wide class of competing hypotheses.

In this paper, some of the tests are excluded from consideration as they show unsatisfactory properties in important cases. Following [3], we excluded the tests of Epstein, Hartley, Deshpande (J_α), and Wong and Wong. Among the tests considered in [4] – H_{n,a} entropy estimator-based test that shows low power and the tests of Henze and Meintanis with statistics \( T_{n,a}^{(1)} \) and \( T_{n,a}^{(2)} \) that show unexpectedly low power in several cases either.

We conducted a comparative analysis of tests from the promising group. In addition, we considered Bolshev’s test for exponentiality proposed for testing hypothesis of exponentiality of observations in several small samples.

II. PROBLEM DEFINITION

Let \( \exp(\theta) \) be exponential distribution with the density \( f(x) = \exp(-x/\theta) / \theta, \ x \geq 0, \ \theta = \lambda^{-1} > 0 \), and \( X_1, \ldots, X_n \) be given independent observations of nonnegative random variate. The composite hypothesis under test is \( H_0: X \) follows \( \exp(\theta) \) under some value of \( \theta \).

In test statistics, we will use scaled observations \( Y_j = X_j / \hat{\theta}_n \) or their transformed values \( Z_j = 1 - \exp(-Y_j) \), \( 1 \leq j \leq n \), where \( \hat{\theta}_n = \sum_{j=1}^{n} X_j / n \) is the maximum likelihood estimator of parameter \( \theta \).

Let us denote order statistics of \( X_1, Y_j, \) and \( Z_j \) as \( X_{(1)}, Y_{(j)}, \) and \( Z_{(j)} \) respectively. Denote:

\[
D_j = (n - j + 1)(X_{(j)} - X_{(j-1)}), \quad 1 \leq j \leq n, \quad X_{(0)} = 0.
\]

III. THEORY

A. Gnedenko’s F-test

Gnedenko’s F-test [3, 5, 6] is designed to test exponentiality against competing hypothesis \( H_j \); distribution has monotone hazard rate. The test statistic is:

\[
Q_k = \sum_{j=1}^{k} D_j / R / \sum_{j=R+1}^{n} D_j / (n - R).
\]

Under true null hypothesis, \( Q_k \) has an F distribution with \( 2R \) and \( 2(n-R) \) degrees of freedom. \( H_0 \) is rejected for both small and large values of \( Q_k \), concluding a decreasing hazard rate in the first case and an increasing – in the second. Our simulation with calculation of estimators of power of Gnedenko’s test show that one should set \( R=[0.3n] \) out of \( R=[0.1n], [0.2n], \ldots, [0.9n] \) to maximize power when testing against hypotheses with monotone hazard rate.
B. Harris’ modification of Gnedenko’s F-test

This test was proposed by Harris [7] and discussed in [3] and [5]. The test statistic
\[
Q^*_R = \frac{\left( \sum_{i=1}^{n} D_i + \sum_{j=K+1}^{R} D_j \right)^2}{2R} \bigg/ \frac{1}{(n-2R)}
\]
follows \(F\)-distribution with \(4R\) and \(2(n-2R)\) degrees of freedom under true null hypothesis. The hypothesis is rejected for both small and large values of \(Q^*_R\).

We obtained that this test has sufficiently high power with respect to competing hypotheses with convex hazard rate and low power with respect to distributions with monotone hazard rate. The simulation conducted have shown that the test reaches its highest power with \(R=0.1n\).

C. Hollander and Proshan’s test

The test of Hollander and Proshan [8, 3] is applied to one-sided alternatives with property “new better than used” (“new worse than used”). This property “may be interpreted as stating that the chance \(F(x)\) that a new unit will survive to age \(x\) is greater (less) than the chance \(\overline{F}(x+y)/\overline{F}(y)\) than an unfailed unit of age \(y\) will survive an additional time \(x\). That is, a new unit has stochastically greater life than a used unit of any age” [8]. The test statistic is:

\[
T = \sum_{i,j \leq k} \psi \left( X_{(i)}, X_{(j)} + X_{(k)} \right), \psi(a,b) = \begin{cases} 1, & a > b, \\ 0, & a \leq b. \end{cases}
\]

The test is two-sided, authors give tables of approximated lower and upper critical values and the following normal approximation:

\[
T^* = \left( T - E[T | H_0] \right) \left( D[T | H_0] \right)^{-1/2},
\]

where

\[
E[T | H_0] = n(n-1)(n-2)/8 \quad \text{and} \quad D[T | H_0] = 1.5n(n-1)(n-2)x \times 2(n-3)(n-4)/2592 + 7(n-3)/432 + 1/48.
\]

D. Gini’s test

This two-sided test with statistic
\[
G_n = \frac{\sum_{i,j \leq k} \left| Y_j - Y_i \right|}{2n(n-1)}
\]
is considered in [9, 3, 4].

The asymptotic distribution of
\[
G_n^* = \left[ 12(n-1) \right]^{-1/2} \left\{ G_n - 1/2 \right\}
\]
is standard normal [9] which, as we found, well describes \(G_n^*\) under \(n \geq 10\). Gini’s test is equivalent to the score test [4] with statistic \(S_n = 2n - 2\sum_{i,j \leq k} \left| Y_j - Y_i \right|\), which is connected to \(G_n\) with expression \((n-1)^{1/2} S_n = 1 - G_n\). We considered the test based on \(G_n\) expressing the latter via \(S_n\) as it has lower computational complexity.

E. Tests based on empirical distribution function

E.1. Kolmogorov’s test

In Kolmogorov’s goodness-of-fit test, the value
\[
D_n = \max_{i < j} \left| \frac{j}{n} - Z_{(i)} \right| \max_{i > j} \left| Z_{(i)} - \frac{j-1}{n} \right|
\]
is used as a measure of difference between empirical distribution and the exponential law.

To decrease the dependence of the Kolmogorov’s statistic on sample volume one should use the statistic with Bolshev’s correction [10]:

\[
K_n = \left( \sqrt{n} \cdot D_n + 1 \right) / \sqrt{6n}.
\]

Fig. 1. Densities of distributions of statistics \(K, CMS, \) and \(AD\) under true null hypothesis.

E.2. Cramer–von Mises–Smirnov’s test

The test statistic of Cramer–von Mises–Smirnov’ test is:

\[
CMS_n = \frac{1}{12n} + \sum_{i,j \leq k} \left( Z_{(i)} - \frac{j-1}{2n} \right)^2.
\]

E.3. Anderson–Darling’s test

The statistic of Anderson–Darling’s goodness-of-fit test for testing a sample for exponentiality is:

\[
AD_n = -n - 2\sum_{i,j \leq k} \left[ \frac{2j-1}{2n} \ln Z_{(i)} + \left( 1 + \frac{2j-1}{2n} \right) \ln \left( 1 - Z_{(i)} \right) \right].
\]

The hypothesis of exponentiality is rejected by either Kolmogorov’s, Cramer–von Mises–Smirnov’s, or Anderson–Darling’s test for large values of statistic.

A good model [11] for distribution of \(K_n\) under true complex null hypothesis and \(n \geq 25\) is gamma distribution \(\gamma(5.1092; 0.0861; 0.295)\) with density

\[
f(x) = \frac{1}{\theta_2^5 \Gamma(\theta_2)} x^{\theta_2 - 1} e^{-(x - \theta_2)/\theta_2}, \ x > \theta_2.
\]
for $CMS_n$ – Johnson’s SB Distribution $Sb(3.3738;0.2145;1.0792;0.011)$ with density

$$f(x) = \frac{\theta_3 \theta_2}{\sqrt{2\pi} (x-\theta_1) (\theta_1 + \theta_2 - x)} \times$$

$$\exp \left\{ -\frac{1}{2} \left( \frac{\theta_2 + \theta_1 \ln \frac{x-\theta_1}{\theta_1 + \theta_2 - x}}{\theta_1 + \theta_2 - x} \right)^2 \right\}, \ x \in [\theta_1, \theta_1 + \theta_2] ,$$

for $AD_n$ – $Sb(3.8386;1.3429;7.500;0.090)$ (see Fig.1).

F. Tests based on a characterization via the mean residual life function

$X$ is distributed exponentially under the assumption $0 < \mu < \infty$ if, and only if $E[X-t | X > t] = \mu$ for each $t > 0$. This is equivalent to $E[\min(X,t)] = \mu F(t)$ for each $t > 0$ and, basing upon this, Baringhaus and Henze [12, 4] proposed Kolmogorov and Cramer–von Mises–Smirnov type statistics.

The Kolmogorov type statistic of Baringhaus-Henze is:

$$K_n = (6n \cdot \bar{K}_n)/\sqrt{n} + 1]/6\sqrt{n} .$$

The Cramer–von Mises–Smirnov statistic of Baringhaus-Henze is:

$$CMS_n = n^{\frac{1}{6}} \left[ 2 - 2 e^{-\min(Y_i, x)} - 2 \min(Y_i, x) \times$$

$$\times \left( e^{Y_i} + e^{-Y_i} \right) + 2 e^{-\max(Y_i, x)} \right] .$$

The results of our simulation show that distributions of these two statistics do not match those given in [12], though one should not be surprised by this fact because the hypothesis under discussion is composite and involves calculation of MLE of scale parameter [13] (see also section IV).

G. Deshpande’s test

The test was proposed in [14] and discussed in [3] for testing exponentiality against competing distributions with increasing hazard rate. The test statistic is calculated by

$$J_b = n(n-1)^{-1} \sum h_b(X_i, X_k) ,$$

where $h_b = \begin{cases} 1, & X_i > bX_k , \\ 0, & \text{otherwise} . \end{cases}$

and the sum is taken for all $1 \leq j,k \leq n, j \neq k$. When nothing is known about competing distribution a priori, one should use two-sided critical values.

Deshpande showed that $n^{1/2} (J_b - M(F))$ has asymptotically normal distribution with $\mu = 0$ and $\sigma^2 = 4\zeta_r$, where

$$M(F) = (b+1)^{-1}$$

$$\zeta_r = \frac{1}{4} \left[ (1 + b + \frac{2(1-b)}{b+1})^2 \right] .$$

H. Cox and Oakes test

The hypothesis under test is rejected for both small and large values of test statistic

$$CO_n = n + \sum_{j=1}^n (1-Y_j) \log Y_j .$$

The normalized statistic $CO_n \sqrt{n}/\pi^{1/2}$ has limit standard normal distribution.

The test with statistic $CO_n$ is consistent against competing distributions with finite mathematical expectation and $E[X \log X - \log X] \neq 1$, provided the latter expectation exists.

I. Klar’s test

The Klar’s test [15, 4] is based upon the integrated distribution function and rejects the hypothesis of exponentiality for large values of statistic

$$KL_{n,a} = \frac{2(3a+2)n}{(2+a)(1+a)}$$

$$-2a \sum_{j=1}^n \exp(-(1+a)Y_j) \frac{2}{n} \sum_{j=1}^n \exp(-aY_j) +$$

$$+2 \sum_{j<k} \left[ a(Y_i - Y_j) - 2 \right] \exp(-aY_j) .$$

The author proposes [15] the use of a combined test that is based upon several statistics $KL_{n,a}$ with different values of $a$ and rejects the hypothesis if at least one of $KL_{n,a}$ tests rejects it. Relying on simulation results, author concludes that the test $KL_{n,a}^{combined}$ (combined of $KL_{n,a}$ and...
be independent random variables. The hypothesis to test is $H_n$: $X_i$ follow exponential distributions with densities $a_i \exp(-a x_i)$ ($x > 0$, $j = 1, n_i$; $i = 1, N$); the values of $a_i$ are unknown and, possibly, different. If $H_n$ is true, the statistics $\zeta_n = \sum_{j=1}^{n_i} X_j / \sum_{j=1}^{n_i} Y_j$ ($r = 1, n_i - 1$) are independent and follow beta distributions with parameters $r$ and 1 [16]. Consequently, statistics $\zeta_n$ ($r = 1, n_i - 1$; $i = 1, N$) are independent and identically uniformly distributed on $[0,1]$. One should apply non-parametric goodness-of-fit tests to test them for uniformity. In this paper, we used Anderson–Darling’s test [17]. Total volume of small samples has a determinative effect on power of the Bolshev’s test, thus we consider single samples without loss of generality.

K. Tests based upon empirical Laplace transform

In these tests, the Laplace transform $\psi(t) = E[\exp(-tX)] = \lambda/(\lambda + t)$ of exponential distribution is estimated by its empirical counterpart $\psi_n(t) = \frac{1}{n} \sum_{j=1}^{n_i} \exp(-t Y_j)$.

K.1. The test of Baringhaus and Henze

In the test of Baringhaus and Henze [18, 4], the fact is used that $\psi$ satisfies differential equation $(\lambda + t) \psi(t) + \psi(t) = 0$, $t \in R$. The test rejects the hypothesis for large values of statistic

$$BH_{n,a} = n^{-1} \sum_{j=1}^{n_i} \left[ \frac{(1 - Y_j)(1 - Y_j)}{Y_j + Y_j + a} - \frac{Y_j + Y_j}{(Y_j + Y_j + a)} \right]^2 + \frac{2Y_j}{(Y_j + Y_j + a)} + \frac{2Y_j}{(Y_j + Y_j + a)}.$$ 

The choice of $a$ is proposed to be made according to a supposed competing hypothesis.

K.2. The test of Henze

The test of Henze [19, 4] rejects the hypothesis of exponentiality for large values of statistic

$$HE_{n,a} = n^{-1} \sum_{j=1}^{n_i} \left( Y_j + Y_j + a \right)^{-1} - \sum_{j=1}^{n_i} \exp(Y_j + a) E(Y_j + a) + n(1 - \exp(a) E(a)),$$

where $E_a(z) = \int_{t=a}^{\infty} r^{-\exp(-t)} dt$ is exponential integral and $a > 0$ is constant.

K.3. The test of Henze and Meintanis

In the L-test of Henze and Meintanis [20, 4], the hypothesis is rejected for large values of statistic $L_{n,a}$. Description of $L_{n,a}$ distribution and tables of percent points for several $a$ are given in [20].

$$L_{n,a} = \frac{n}{\sum_{j=1}^{n_i} \left( Y_j + Y_j + a + a \right)^{3} - 2 \sum_{j=1}^{n_i} (Y_j + a)^{2} + n a^2}.$$

L. Tests based upon empirical characteristic function

L.1. W-tests of Henze and Meintanis

In the $W$-tests of Henze and Meintanis [21, 4], the null hypothesis is rejected for large values of statistics:

$$W_{n,a}^{(1)} = \frac{\sqrt{n}}{2a \sqrt{\pi}} \sum_{j=1}^{n_i} \left[ \frac{1}{a^2 + (Y_j - Y)^2} - \frac{1}{a^2 + (Y_j + Y)^2} \right]$$

$$- \frac{4(Y_j + Y)}{a^2 + (Y_j + Y)^2} + \frac{2a^2 - 6(Y_j - Y)^2}{a^2 + (Y_j - Y)^2} + \frac{2a^2 - 6(Y_j + Y)^2}{a^2 + (Y_j + Y)^2}.$$ 

$$W_{n,a}^{(2)} = \frac{n}{\sqrt{4a^2}} \sum_{j=1}^{n_i} \left[ 1 + \frac{2a - (Y_j - Y)^2}{4a^2} \right] \exp \left[ - \frac{(Y_j - Y)^2}{4a} \right] +$$

$$+ \frac{2a - (Y_j + Y)^2}{4a^2} \exp \left[ - \frac{(Y_j + Y)^2}{4a} \right].$$

L.2. Test for exponentiality of Epps and Pulley

As $n \to \infty$ the statistic of the test of Epps and Pulley [4]

$$EP_{n} = \left( 48n \right)^{1/2} \frac{1}{2} \sum_{j=1}^{n} \exp \left[ - (Y_j - Y_{j-1}) \right]^{-1/2}$$ 

is described by standard normal distribution; the null hypothesis is rejected for large values of $|EP_{n}|$. The test is consistent against competing distributions with monotone hazard rate, absolutely continuous CDF $F(x)$, $F(0) = 0$, and $0 < \mu < \infty$.

IV. EXPERIMENTAL RESULTS

Some of the authors give normalizing transformations for test statistics, what makes it possible to apply standard normal law to normalized statistic to compute $p$-values.
while testing the hypothesis. In practice, such asymptotical results may turn to be unacceptable for samples of finite volume as a consequence of significant difference between distribution of specific statistic and its asymptotical model.

We used the methodology of statistical simulation [22] to verify how close actual distributions of statistics fit to corresponding theoretical models. The normalizing transformations were applied to statistics $T$, $G$, $J$, and $CO$ when computing empirical distributions of test statistics under true null hypothesis. The results are based on 16'600 simulations, the true distribution was exponential with $\theta = 1$: $F(x) = 1 - \exp(-x)$. The samples obtained were tested for fit with corresponding limit distributions by classical Kolmogorov’s test. The $p$-values obtained in testing the simple hypothesis are given in Table I.

The results are following.

Application of limit distributions in the tests $Q_R$, $Q_{Kw}$, $G$, $K$, CMS, $AD$, $B$ is correct and makes it possible to accurately estimate the $p$-value.

Tests $K^*$ and CMS* are not delivered from the influence of sample volume on distribution of statistic. For $n \geq 20$, Johnson’s SB distribution $Sb(2.1275;1.6849;2.5437;0.26888)$ can serve as model for $K^*$ and $Sb(2.756;0.98223;1.8645;0.01602)$ – for CMS*. When $n = 10$ the use of these models leads to an underestimated $p$-value by $K^*$ test and an overestimated $p$-value by CMS*.

The normal approximation of distribution of statistic HP can be used only with limitations. Under $n \leq 300$, computation of percent point tables would be the best choice. The use of asymptotical model is reasonable under $n \geq 400$.

In the test with statistic $J_{0.5}$, application of normal approximation do not lead to significant errors under $n \geq 50$; in tests with statistics $EP$ and $CO$ – under $n \geq 100$.

V. DISCUSSION OF RESULTS

We compared the power of tests for relatively small sample volumes $n = 20$ and $n = 50$. Empirical distributions of test statistics under either true null hypothesis or competing hypotheses were found by 1'660'000 simulations. Null (exponential) distribution is characterized by constant hazard rate, thus we considered competing distributions that belong to three classes: with increasing, decreasing, and non-monotone hazard rates:

- Weibull $W(\theta)$ with $f(x) = \theta x^{\theta-1} \exp(-x^\theta)$;
- gamma $\Gamma(\theta)$ with $f(x) = \Gamma(\theta)^{-1} x^{\theta-1} \exp(-x)$;
- beta $B(\theta_o, \theta_i)$ with $f(x) = B(\theta_o, \theta_i)^{-1} x^{\theta_o-1} (1-x)^{\theta_i-1}$;

- uniform $U(0,1)$ on $[0,1]$;
- lognormal $LN(\theta)$ with $f(x) = \left(\theta x^{x} \pi e\right)^{-1} \exp\left(-\left(\ln x\right)^2/2\theta^2\right)$;
- half-normal $HN - f(x) = (2/\pi)^{1/2} \exp\left(-x^2/2\right)$.

Distributions with increasing hazard rates are $W(\theta)$ and $\Gamma(\theta)$ ($\theta > 1$), $U(0,1)$, $HN$, $B(1,2)$, $B(2,1)$; decreasing – $W(\theta)$ and $\Gamma(\theta)$ ($\theta < 1$); non-monotone – $LN$, $B(0.5,1)$.

When computing critical values of statistics and estimators of power we assumed no prior knowledge of type of competing hypothesis. Therefore, we used two-sided critical regions in those tests that have a choice between left-sided and right-sided critical regions.

The estimators of power of tests with respect to different competing distributions with increasing, decreasing, and non-monotone hazard rates are given in Tables II, III, and IV respectively.

The tests with statistics $BH$ and $HE$ behave alike (this fact was mentioned in [4]), therefore below we will mention only the test with statistic $BH$. The choice of $a = 0.5$ (and, correspondingly, $BH_{a,\alpha}$) provides higher power compared to other values of $a$. In the $L$-test, statistic $L_4$ would be a reasonable choice in general case, in $W$-tests – statistic $W_{1.0}^{(1)}$, in $KL$-test – $KL^{ne}$, obviously.

The following drawbacks should be mentioned in case of competing hypotheses with increasing failure rate (see Table II). Under $n = 20$, the test of Bolshev is biased with respect to $W(1.2)$, $\Gamma(1.5)$, $HN$, and $B(1.2)$ (i.e., its power is less than probability of type I error $\alpha = 0.05$); the test $L_{0.1}$ is biased with respect to the same distributions and $W(1.4)$; the test $W_{2.0}^{(2)}$ is biased with respect to $W(1.2)$.

The test $Q_{2.5}$ shows remarkably low power with respect to competing distributions with decreasing failure rate (see Table III).

In case of competing laws with non-monotone failure rate, the tests $W_{2.0}^{(0)}$ and $BH_{a}$ are biased with respect to $B(0.5,1)$; the test $L_{0.1}$ – with respect to $LN(1)$ and $LN(0.8)$.

VI. CONCLUSION

Obviously, among the all tests studied, we cannot unambiguously choose a test with the highest power with respect to every considered competing hypothesis. It is as well unrealistic to place the tests in some unconditional order, e.g., descending by power.
In the same time, it is possible to select groups of tests equally promising in case of suggestion of certain kind of alternative.

Thus with respect to competing distributions with both increasing and decreasing failure rates, the tests of Cox and Oakes (CO), Anderson and Darling (AD), Henze and Meintanis ($L_{\alpha}$ and $W_{\alpha}^{(2)}$), Baringhaus-Henze ($BH_{\alpha}$), and Henze ($HE_{\alpha}$) show stably high power.

The tests of Harris ($Q_{1,1}$) and Anderson and Darling (AD) possess high power with respect to alternatives with non-monotone hazard rates.

It is undesirable to use the tests of Harris ($Q_{1,1}$), Bolshev ($B$), Henze and Meintanis ($L_{\alpha}$ and $W_{\alpha}^{(2)}$), Baringhaus and Henze ($BH_{\alpha}$), and Henze ($HE_{\alpha}$) under condition of small sample size or without specifying a concrete alternative (as a result of possible bias).

In a problem of choice of the most powerful test against given specified alternative beyond the ones considered in this paper, one should conduct a power research by similar methodology (and try different values of “fine-tuning” parameters in tests that have such). Of course, in such a research, knowledge of hazard rate function of the alternative should be taken into account.

The Bolshev’s test ($B$) possess sufficiently high power with respect to laws with decreasing hazard rates but is inferior to the other tests in cases of other alternatives. One should keep in mind that the main advantage of the test is the approach that makes it possible to test the hypothesis of exponentiality of a set of small samples.

REFERENCES


### Table I

**P-Values in Testing Goodness-of-Fit of Test Statistic Distributions with Corresponding Theoretical Models**

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### Table II

**Power of tests for exponentiality with respect to competing hypotheses with increasing failure rate ×1000 (n=20, α=0.05).**

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<th>$U(1.4)$</th>
<th>$U(1.5)$</th>
<th>$U(1.2)$</th>
<th>$U(1.4)$</th>
<th>$U(1.5)$</th>
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<th>$U(1.4)$</th>
<th>$U(1.5)$</th>
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</table>

---

**Legend:**

- $B$: The failure rate is increasing at the rate of $B$.
- $W$: The power of the test is $W$.
- $U$: The power of the test is $U$.

**Note:**

- The powers are given in percentages.
- The values are rounded to the nearest whole number.
- The table includes the power values for different failure rates and test statistics.
**TABLE III**

Power of tests for exponentiality with respect to competing hypotheses with decreasing failure rate $\times 1000$ ($n=20$, $\alpha=0.05$).

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<th>$\Gamma(0.7)$</th>
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<th>$\Gamma(0.5)$</th>
<th>$\Gamma(0.4)$</th>
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<td>$J_{53}$</td>
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<td>564</td>
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</tr>
<tr>
<td>$EP$</td>
<td>200</td>
<td>236</td>
<td>543</td>
<td>759</td>
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<td>$G$</td>
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<td>239</td>
<td>547</td>
<td>759</td>
</tr>
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<td>$Q_{0.5}$</td>
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<td>193</td>
<td>567</td>
<td>787</td>
</tr>
<tr>
<td>$Q'_{0.5}$</td>
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<td>165</td>
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<td>811</td>
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<td>$K$</td>
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<td>173</td>
<td>470</td>
<td>706</td>
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<tr>
<td>$K'$</td>
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<td>134</td>
<td>380</td>
<td>617</td>
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<td>$CMS$</td>
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<td>525</td>
<td>756</td>
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<td>523</td>
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<td>279</td>
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**TABLE IV**

Power of tests for exponentiality with respect to competing hypotheses with non-monotone failure rate $\times 1000$ ($n=20$, $\alpha=0.05$).

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