STATISTIC DISTRIBUTIONS OF THE NONPARAMETRIC GOODNESS-OF-FIT TESTS IN TESTING HYPOTHESES RELATIVE TO BETA-DISTRIBUTIONS*

Stainislav B. Lemeshko[1] and Boris Yu. Lemeshko[2]

[1] Novosibirsk State Technical University
630092 Novosibirsk, Russia
(e-mail: skyer@mail.ru)
[2] (e-mail: headrd@fpm.ami.nstu.ru)

Abstract. The paper considers composite hypotheses testing relative to the beta-distributions of the I, II and III type. In case of estimating of law parameters, according to the same simple statistic distributions of the nonparametric goodness-of-fit tests depend on the specific values of the beta-distributions of the I, II and III type form parameters. In this paper, distribution models and tables of the percentage points of the Kolmogorov, the Cramer-Mises-Smirnov, the Anderson-Darling statistics are constructed for different value combinations of two beta-distributions form parameters (depending on the type and number of parameters estimated).
Keywords: goodness-of-fit test, composite hypotheses testing, Kolmogorov test, Cramer-Mises-Smirnov test, Anderson-Darling test.

In composite hypotheses testing of the form $H_0: F(x) \in \{F(x,\theta), \theta \in \Theta\}$, when the estimate $\hat{\theta}$ of the scalar or vector distribution parameter $F(x,\theta)$ is calculated by the same sample, statistic distributions $G(S\,|H_0\,)$ of the nonparametric goodness-of-fit tests depend on a number of factors: the form of the observed law $F(x,\theta)$ corresponding to the true hypothesis H_0; the type of the parameter estimated; the number of parameters to be estimated; the method of parameter estimation. Sometimes, statistic distributions $G(S\,|H_0\,)$ depend on concrete values of the parameter or parameters. For example, in the case of gamma-distributions [R 50.1.037-2002, 2002] and beta-distributions families, the laws of statistic distributions $G(S\,|H_0\,)$ depend on specific values of gamma- and beta-distributions form parameters.

Since the publication of work [Kac *et al.*, 1955], many researchers were engaged in the problems of nonparametric goodness-of-fit tests in composite hypotheses testing, for example [Durbin, 1955], [Martinov, 1978], [Tyurin, 1984], [Stephens, 1970], [Pearson and Hartley, 1978], [Stephens, 1974], [Chandra *et al.*, 1974]. The statistic distributions were also investigated in

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our works [Lemeshko, 1998], [Lemeshko and Postovalov, 2001], [Lemeshko and Maklakov, 2004] by methods of statistical modeling.

In this paper, statistic $G(S\,|H_0\,)$ distributions of Kolmogorov test, ω^2 Cramer-Mises-Smirnov test, Ω^2 Anderson-Darling test were investigated in testing of hypotheses relative to the families of beta-distributions of the I and II type in case of applying maximum likelihood methods for parameter estimation of these laws. Beta-distribution of the I-th type has the density function

$$f(x) = B_I(\theta_0, \theta_1) = \frac{1}{\theta_2 B(\theta_0, \theta_1)} \left(\frac{x}{\theta_2}\right)^{\theta_0 - 1} \left(1 - \frac{x}{\theta_2}\right)^{\theta_1 - 1}, \tag{1}$$

were $B(\theta_0, \theta_1) = \Gamma(\theta_0)\Gamma(\theta_1)/\Gamma(\theta_0 + \theta_1)$ is beta-function, form parameters $\theta_0, \theta_1 \in (0, \infty)$, scale parametr $\theta_2 \in (0, \infty), x \in [0, \theta_2]$.

The density function of beta-distribution II-th type described by expression

$$f(x) = B_{II}(\theta_0, \theta_1) = \frac{1}{\theta_2 B(\theta_0, \theta_1)} \frac{[x/\theta_2]^{\theta_0 - 1}}{[1 + x/\theta_2]^{\theta_0 + \theta_1}}, \tag{2}$$

were $x \in [0, \infty)$. F - distribution of Fisher is a special case of beta-distribution II-th type.

Statistic of Kolmogorov test [Bolshev and Smirnov, 1983] was considered with Bolshev correction [Bolshev L.N., 1963] of the form

$$S_K = \frac{6nD_n + 1}{6\sqrt{n}}, \tag{3}$$

where

$$D_n = \max(D_n^+, D_n^-),$$

$$D_n^+ = \max_{1 \le i \le n} \left\{ \frac{i}{n} - F(x_i, \theta) \right\}, D_n^- = \max_{1 \le i \le n} \left\{ F(x_i, \theta) - \frac{i-1}{n} \right\},$$

n is the sample size, $x_1, x_2, \ldots, x_n$ are sample values arranged in increasing ordered fashion. The distribution of statistic (3) in testing simple hypotheses obeys the Kolmogorov distribution law $K(S)$.

A statistic of Cramer-Mises-Smirnov test ω^2 is of the form [Bolshev and Smirnov, 1983]

$$S_\omega = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F(x_i, \theta) - \frac{2i-1}{2n} \right\}^2, \tag{4}$$

and a statistic of Ω^2 Anderson-Darling test is of the form [Bolshev and Smirnov, 1983] -

$$S_\Omega = -n - 2\sum_{i=1}^{n} \left\{ \frac{2i-1}{2n} \ln F(x_i, \theta) + \left(1 - \frac{2i-1}{2n}\right) \ln(1 - F(x_i, \theta)) \right\}. \tag{5}$$

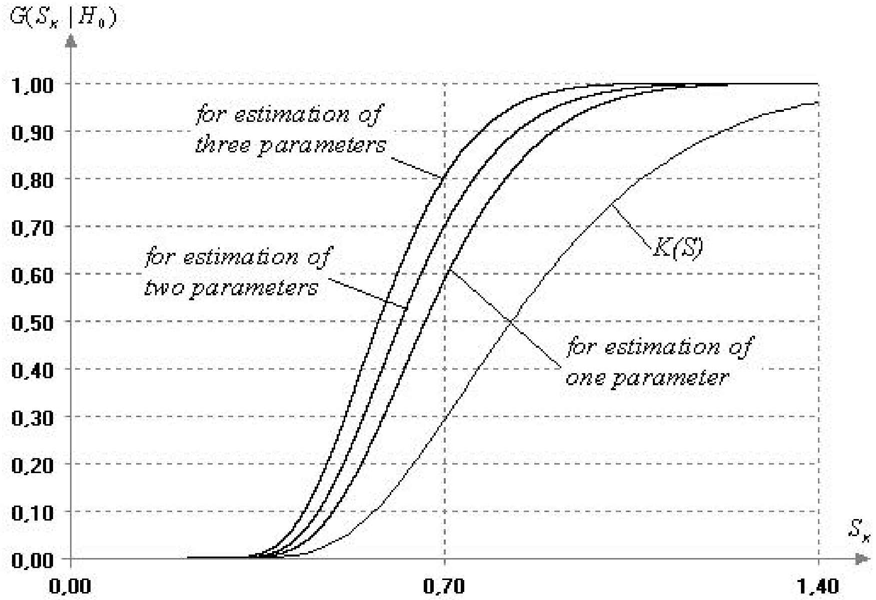

Fig. 1. Dependence of Kolmogorov statistics (3) distributions on the number of estimated parameters of $B_{II}(\theta_0, \theta_1)$-distribution (for estimation: θ_0 or θ_1; θ_0 and θ_1; θ_0,θ_1 and θ_2) for the $\theta_0 = \theta_1 = 10$.

In testing a simple hypothesis, statistic (4) obeys the distribution $a1(S)$, and statistic (5) obeys the distribution $a2(S)$[Bolshev and Smirnov, 1983].

Figure 1 illustrates the changing Kolmogorov statistic distributions (3) depending on the number of estimated parameters of $B_{II}(\theta_0, \theta_1)$ -distribution: one of the parameters of the form θ_0 or θ_1; two parameters of the form θ_0 and θ_1 three parameters θ_0, θ_1 and θ_2. Statistic (3) distributions $G(S_K|H_0)$ are shown for case $\theta_0 = 10$ and $\theta_1 = 10$. For comparison, the Figure 1 presents Kolmogorov distribution $K(S)$, which is limiting distribution in simple hypotheses testing.

Figures 2 and 3 show the changing of statistic distributions (4) and (5) in relation to the values of parameters θ_0 and θ_1 $B_I(\theta_0, \theta_1)$-distribution in estimating two parameters or one of the parameters θ_0, θ_1. For values θ_0 and $\theta_1 > 10$ statistics (4) distributions $G(S_\omega|H_0)$ and $G(S_\Omega|H_0)$, statistics (5) practically cease to change. It's typical for distributions of statistic (3) as well.

It should be emphasized that in goodness-of-fit hypotheses testing with $B_{II}(\theta_0, \theta_1)$-distribution for statistics (3)-(5) there are distributions $G(S|H_0)$ similar to $B_I(\theta_0, \theta_1)$-distributions.

Usually statistic distributions $G(S|H_0)$ of (3) Kolmogorov,(4) Cramer-Mises-Smirnov and (5) Anderson-Darling are well approximated by one of the following distribution families: gamma-distributions with the density function

$$\gamma(\theta_0,\theta_1,\theta_2) = \frac{1}{\theta_1^{\theta_0}\Gamma(\theta_0)}\,(x-\theta_2)^{\theta_0-1}\,e^{-(x-\theta_2)/\theta_1};$$

Sb-Johnson distributions with the density function

$$Sb(\theta_0,\theta_1,\theta_2,\theta_3) = \frac{\theta_1\theta_2}{(x-\theta_3)(\theta_2+\theta_3-x)} exp\left\{-\frac{1}{2}\left[\theta_0-\theta_1 ln\frac{x-\theta_3}{\theta_2+\theta_3-x}\right]^2\right\};$$

Sl-Johnson distributions with the density function

$$Sl(\theta_0,\theta_1,\theta_2,\theta_3) = \frac{\theta_1}{\sqrt{2\pi}(x-\theta_3)} exp\left\{-\frac{1}{2}\left[\theta_0+\theta_1 ln\frac{x-\theta_3}{\theta_2}\right]^2\right\};$$

beta-distributions of the III-th type with the density function

$$B_{III}(\theta_0,\theta_1,\theta_2,\theta_3,\theta_4) = \frac{\theta_2^{\theta_0}}{\theta_3 B(\theta_0,\theta_1)}\frac{\left(\frac{(x-\theta_4)}{\theta_3}\right)^{\theta_0-1}\left(1-\frac{(x-\theta_4)}{\theta_3}\right)^{\theta_1-1}}{\left(1+(\theta_2-1)\frac{(x-\theta_4)}{\theta_3}\right)^{\theta_0+\theta_1}},$$

where the parameters of the form $\theta_0,\theta_1,\theta_2 \in (0,\infty)$, scale parameter $\theta_3 \in (0,\infty)$, $x \in [\theta_4, \theta_4+\theta_3]$.

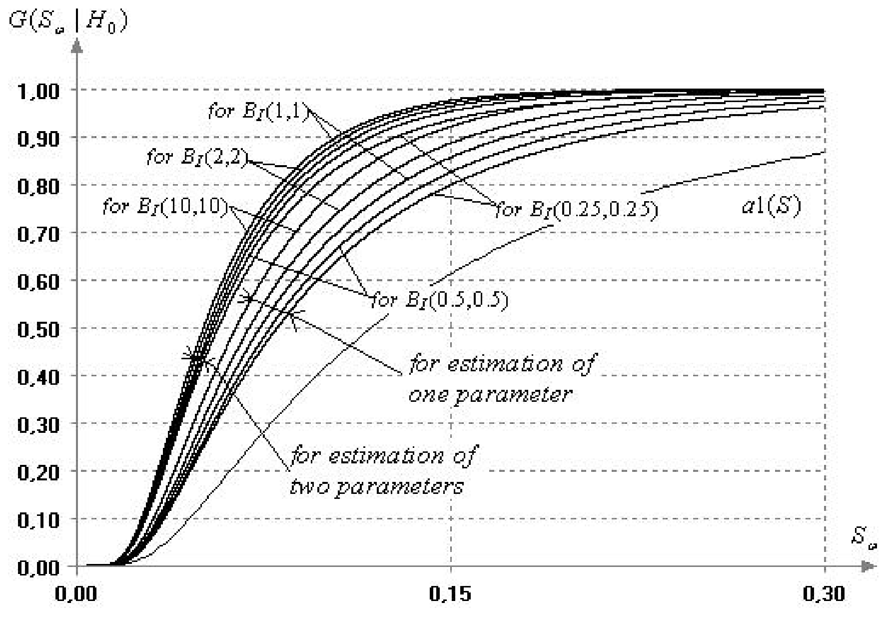

Fig. 2. Dependence of Cramer-Mises-Smirnov statistics (4) distributions on the parameters values θ_0 and θ_1 of $B_I(\theta_0,\theta_1)$-distribution (for estimating two parameters or one of tht parameters θ_0, θ_1)

In this research tables of percentage points and statistic distribution models were obtained for the nonparametric goodness-of-fit tests in testing composite hypotheses subject to beta-distributions of the I-th II-th type using the maximum likelihood estimate (MLE). Some of this results are presented in table 1. It is interesting that in estimation of two parameters of the form with the increase in values of one of these dependence on values of this parameter practically disappears. For example, for the case where $\theta_0 = 1$ and increase in values θ_1 from 2 to higher gamma-distribution $\gamma(6,2541;0,0622;0,2640)$ is a good model for distribution $G(S_K|H_0)$ of Kolmogorov statistics (3), Sl-Johnson distribution $Sl(0,8373;1,8500;0,4800;0,0450)$ is a good model for distribution $G(S_\Omega|H_0)$ of Anderson-Darling. In the same situation, distribution $G(S_\omega|H_0)$ of Cramer-Mises-Smirnov statistics (4) is well approximated by Sl-Johnson and normal laws composite - $0,945 \times Sl(2,1117;1,5484;0,1740;0,0075) + 0,055 \times N(0,0870;0,0297)$.

The results given in Table 1, are useful in composite hypotheses testing relative to beta-distributions of the III type for estimation only θ_0 and θ_1 parameters of this laws.

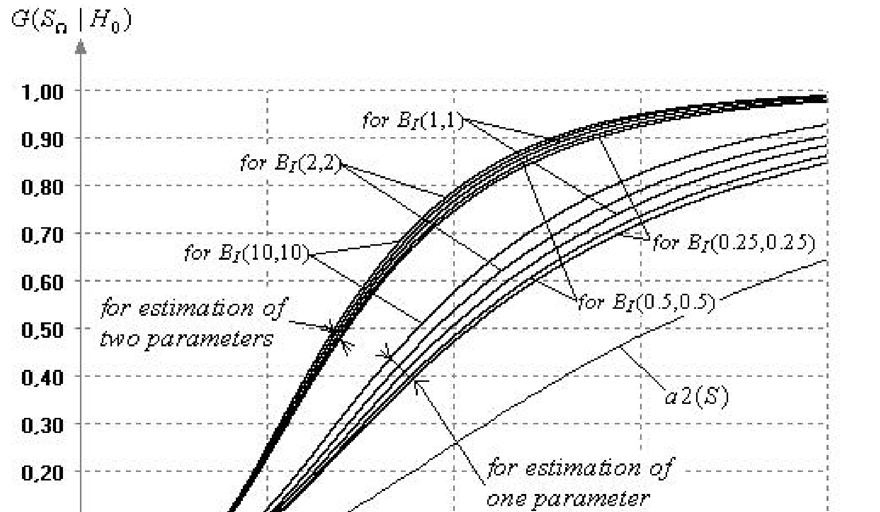

Fig. 3. Dependence of Anderson-Darling statistics (5) distributions of a specific parameters values θ_0 and θ_1 of $B_I(\theta_0, \theta_1)$-distribution (for estimation two parameters or one of parameters θ_0, θ_1)

Table 1. Percentage points and distribution of statistic (3), (4), (5) in composite hypotheses testing and calculating MLE of one or two parameters of the form the I and II type beta-distributions

For Kolmogorov statistic					
For estimation one of parameters θ_0 or θ_1					
θ_0	θ_1	Percentage points			Statistic distribution model
		0.9	0.95	0.99	
0.25	0.25	1.058	1.17	1.398	$B_{III}(6.8597; 5.1140; 4.5522; 1.9581; 0.2803)$
0.5	0.5	1.03	1.135	1.348	$B_{III}(6.6547; 5.0791; 4.0459; 1.7722; 0.2827)$
0.75	0.75	1.01	1.113	1.314	$B_{III}(6.3949; 5.6417; 3.5771; 1.7799; 0.2833)$
1	1	0.994	1.091	1.289	$B_{III}(7.3246; 5.4328; 4.0112; 1.7102; 0.2702)$
2	2	0.957	1.046	1.233	$B_{III}(5.9642; 5.6154; 2.9990; 1.5292; 0.2885)$
3	3	0.939	1.025	1.202	$B_{III}(8.3807; 5.1719; 4.1359; 1.4690; 0.2601)$
4	4	0.928	1.012	1.181	$B_{III}(7.2512; 4.8654; 3.6112; 1.3522; 0.2748)$
5	5	0.921	1.004	1.171	$B_{III}(6.5326; 5.3666; 3.1539; 1.3856; 0.2838)$
6	6	0.915	0.997	1.161	$B_{III}(7.6098; 5.6551; 3.4763; 1.4386; 0.2652)$
7	7	0.912	0.993	1.155	$B_{III}(5.1333; 5.9954; 2.3336; 1.3716; 0.3032)$
8	8	0.909	0.988	1.15	$B_{III}(6.4544; 5.9324; 2.8642; 1.4047; 0.2798)$
9	9	0.907	0.986	1.142	$B_{III}(6.2320; 5.8571; 2.7868; 1.3812; 0.2839)$
10	10	0.905	0.984	1.142	$B_{III}(8.0358; 5.2786; 3.6801; 1.3578; 0.2615)$
For estimation one of parameters θ_0 and θ_1					
θ_0	θ_1	Percentage points			Statistic distribution model
		0.9	0.95	0.99	
0.25	0.25	0.911	1.004	1.208	$B_{III}(6.2931; 6.1562; 3.7837; 1.7288; 0.2826)$
0.5	0.5	0.887	0.97	1.141	$\gamma(5.7078; 0.0687; 0.2744)$
0.75	0.75	0.877	0.957	1.122	$B_{III}(8.7698; 5.4387; 4.4166; 1.4233; 0.2515)$
1	1	0.869	0.948	1.11	$\gamma(6.0588; 0.0641; 0.2695)$
2	2	0.854	0.929	1.083	$B_{III}(6.9450; 5.6772; 3.3264; 1.3184; 0.2709)$
3	3	0.849	0.924	1.076	$B_{III}(8.0883; 5.3028; 3.9525; 1.2880; 0.2608)$
4	4	0.846	0.921	1.071	$B_{III}(7.1541; 5.9954; 3.3447; 1.3508; 0.2664)$
5	5	0.844	0.918	1.069	$B_{III}(6.4824; 5.5411; 3.0032; 1.2364; 0.2748)$
6	6	0.843	0.916	1.067	$B_{III}(6.0438; 6.1303; 2.7738; 1.3066; 0.2798)$
7	7	0.84	0.916	1.066	$B_{III}(6.4246; 5.7070; 2.9437; 1.2503; 0.2744)$
8	8	0.841	0.913	1.066	$B_{III}(7.3916; 5.4188; 3.5829; 1.2627; 0.2683)$
9	9	0.84	0.913	1.059	$B_{III}(7.5935; 6.2434; 3.4384; 1.3711; 0.2602)$
10	10	0.84	0.914	1.062	$B_{III}(7.3966; 5.7184; 3.5120; 1.3084; 0.2651)$
For Cramer-Mises-Smirnov statistic					
For estimation one of parameters θ_0 or θ_1					
θ_0	θ_1	Percentage points			Statistic distribution model
		0.9	0.95	0.99	
0.25	0.25	0.21	0.273	0.433	$B_{III}(4.143; 2.689; 42.948; 1.988; 0.009)$
0.5	0.5	0.192	0.247	0.384	$B_{III}(4.352; 2.723; 29.754; 1.286; 0.009)$
0.75	0.75	0.182	0.232	0.357	$B_{III}(4.440; 2.556; 20.019; 0.790; 0.009)$
1	1	0.174	0.221	0.334	$B_{III}(4.223; 3.025; 19.991; 0.974; 0.009)$
2	2	0.158	0.197	0.294	$B_{III}(5.319; 2.744; 22.232; 0.714; 0.009)$
3	3	0.15	0.187	0.276	$B_{III}(4.244; 3.346; 15.550; 0.768; 0.009)$
4	4	0.147	0.182	0.265	$B_{III}(4.657; 3.144; 14.774; 0.623; 0.008)$
5	5	0.144	0.179	0.26	$B_{III}(3.950; 3.532; 13.323; 0.725; 0.010)$
6	6	0.142	0.177	0.255	$B_{III}(4.087; 3.376; 12.456; 0.627; 0.009)$
7	7	0.142	0.175	0.251	$B_{III}(3.710; 3.568; 11.878; 0.681; 0.010)$
8	8	0.14	0.173	0.25	$B_{III}(3.884; 3.392; 11.710; 0.611; 0.010)$
9	9	0.14	0.173	0.25	$B_{III}(4.419; 3.353; 14.121; 0.641; 0.009)$
10	10	0.139	0.172	0.248	$B_{III}(4.277; 3.450; 13.006; 0.631; 0.009)$

Table 2. Percentage points and distribution of statistic (3), (4), (5) in composite hypotheses testing and calculating MLE of one or two parameters of the form the I and II type beta-distributions. Continuation of Table 1.

For Cramer-Mises-Smirnov statistic					
For estimation one of parameters θ_0 and θ_1					
θ_0	θ_1	Percentage points			Statistic distribution model
		0.9	0.95	0.99	
0.25	0.25	0.129	0.165	0.267	$B_{III}(4.017; 3.864; 28.496; 1.380; 0.009)$
0.5	0.5	0.119	0.148	0.217	$B_{III}(5.619; 3.214; 19.551; 0.561; 0.007)$
0.75	0.75	0.116	0.143	0.209	$B_{III}(4.392; 3.703; 15.289; 0.632; 0.009)$
1	1	0.113	0.139	0.201	$B_{III}(4.878; 3.391; 12.997; 0.453; 0.008)$
2	2	0.109	0.132	0.19	$B_{III}(4.274; 3.912; 11.443; 0.505; 0.009)$
3	3	0.107	0.131	0.187	$B_{III}(4.432; 3.890; 12.071; 0.503; 0.009)$
4	4	0.107	0.13	0.186	$B_{III}(4.626; 3.442; 10.879; 0.388; 0.008)$
5	5	0.106	0.129	0.182	$B_{III}(5.187; 3.435; 12.058; 0.388; 0.007)$
6	6	0.106	0.129	0.182	$B_{III}(4.550; 3.598; 11.115; 0.417; 0.008)$
7	7	0.105	0.128	0.182	$B_{III}(4.784; 3.556; 11.311; 0.402; 0.008)$
8	8	0.105	0.128	0.182	$B_{III}(4.317; 3.406; 10.160; 0.374; 0.009)$
9	9	0.105	0.127	0.18	$B_{III}(4.426; 3.789; 11.165; 0.449; 0.008)$
10	10	0.105	0.128	0.181	$B_{III}(5.929; 3.738; 15.611; 0.478; 0.006)$
For Anderson-Darling statistic					
For estimation one of parameters θ_0 or θ_1					
θ_0	θ_1	Percentage points			Statistic distribution model
		0.9	0.95	0.99	
0.25	0.25	1.185	1.501	2.271	$B_{III}(4.7998; 3.0039; 23.1645; 6.6020; 0.0822)$
0.5	0.5	1.131	1.425	2.137	$B_{III}(4.9816; 3.3050; 22.1179; 6.6710; 0.0762)$
0.75	0.75	1.093	1.362	2.025	$B_{III}(5.5470; 2.9940; 19.1035; 4.6460; 0.0741)$
1	1	01.??	1.311	1.934	$B_{III}(4.5154; 3.5339; 15.5727; 5.3919; 0.0818)$
2	2	0.991	1.219	1.776	$B_{III}(4.9038; 3.2305; 13.9134; 3.8897; 0.0827)$
3	3	0.958	1.179	1.707	$B_{III}(4.6551; 3.7018; 13.4390; 4.4449; 0.0820)$
4	4	0.945	1.155	1.652	$B_{III}(5.4124; 3.6231; 14.5538; 4.1091; 0.0719)$
5	5	0.935	1.142	1.638	$B_{III}(4.9905; 3.7778; 13.6025; 4.2775; 0.0759)$
6	6	0.927	1.132	1.619	$B_{III}(4.9358; 3.7760; 13.3678; 4.2059; 0.0776)$
7	7	0.923	1.126	01.??	$B_{III}(4.3926; 3.8138; 11.7944; 4.1217; 0.0870)$
8	8	0.918	01.??	1.589	$B_{III}(5.0646; 3.7081; 12.8357; 3.8722; 0.0760)$
9	9	0.916	1.115	1.588	$B_{III}(4.5928; 3.6144; 11.2996; 3.6085; 0.0834)$
10	10	0.912	1.113	1.587	$B_{III}(4.9414; 3.8613; 12.9902; 4.1448; 0.0759)$
For estimation one of parameters θ_0 and θ_1					
θ_0	θ_1	Percentage points			Statistic distribution model
		0.9	0.95	0.99	
0.25	0.25	0.696	0.844	1.201	$B_{III}(5.6142; 3.8769; 13.8704; 2.9981; 0.0687)$
0.5	0.5	0.68	0.823	1.163	$B_{III}(6.7982; 3.7476; 15.9038; 2.7538; 0.0634)$
0.75	0.75	0.671	0.809	1.144	$B_{III}(5.6255; 4.4234; 12.9940; 3.1863; 0.0661)$
1	1	0.662	0.797	1.107	$B_{III}(6.0913; 3.8879; 11.6826; 2.3628; 0.0653)$
2	2	0.647	0.772	1.072	$B_{III}(5.1445; 4.4750; 9.8938; 2.6415; 0.0704)$
3	3	0.644	0.766	1.063	$B_{III}(5.2646; 4.4997; 10.4705; 2.7149; 0.0701)$
4	4	0.641	0.765	1.058	$B_{III}(5.6445; 3.8763; 9.6984; 2.0719; 0.0678)$
5	5	0.637	0.762	1.052	$B_{III}(7.1551; 3.8144; 12.1629; 2.0749; 0.0567)$
6	6	0.639	0.761	1.050	$B_{III}(5.7710; 4.1758; 10.8466; 2.4035; 0.0660)$
7	7	0.636	0.76	1.049	$B_{III}(4.6915; 4.0848; 8.0924; 2.1196; 0.0777)$
8	8	0.636	0.76	1.048	$B_{III}(5.6450; 3.8687; 10.1194; 2.1099; 0.0700)$
9	9	0.636	0.759	1.047	$B_{III}(5.5236; 4.1458; 9.8416; 2.2648; 0.0678)$
10	10	0.635	0.759	1.046	$B_{III}(6.7183; 4.3985; 12.6458; 2.6038; 0.0556)$

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