Improvement of statistic distribution models of the nonparametric goodness-of-fit tests in testing composite hypotheses♣

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Abstract. In composite hypotheses testing, when the estimate of the scalar or vector parameter of the law of distribution of probabilities is calculated by the same sample, the nonparametric goodness-of-fit Kolmogorov, Cramer-Mises-Smirnov, Anderson-Darling tests lose the free distribution property. In composite hypotheses testing, the conditional distribution law of the statistic is affected by a number of factors: the form of the observed law of distribution of probabilities corresponding to the true checked hypothesis; the type of the parameter estimated and the amount of parameters to be estimated; sometimes, it is a specific value of the parameter (e.g., in the case of gamma-distribution and beta-distribution families); the method of parameter estimation. In this paper we present more precise results (tables of percentage points and statistic distribution models) for the nonparametric goodness-of-fit tests in testing composite hypotheses using the maximum likelihood estimate (MLE) for some laws of distribution of probabilities.
Keywords: goodness-of-fit test, composite hypotheses testing, Kolmogorov test, Cramer-Mises-Smirnov test, Anderson-Darling test.

In composite hypotheses testing of the form H_0: $F(x) \in \{F(x,\theta), \theta \in \Theta\}$, when the estimate $\hat{\theta}$ of the scalar or vector distribution parameter $F(x,\theta)$ is calculated by the same sample, the nonparametric goodness-of-fit Kolmogorov, ω^2 Cramer-Mises-Smirnov, Ω^2 Anderson-Darling tests lose the free distribution property.

The value

$$D_n = \sup_{|x|<\infty} \left| F_n(x) - F(x,\theta) \right| ,$$

where $F_n(x)$ is the empirical distribution function, n is the sample size, is used in Kolmogorov test as a distance between the empirical and theoretical laws. In testing hypotheses, a statistic with Bolshev [Bolshev and Smirnov, 1983] correction of the form

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$$S_K = \frac{6nD_n + 1}{6\sqrt{n}}, \qquad (1)$$

where $D_n = \max(D_n^+, D_n^-)$,

$$D_n^+ = \max_{1 \le i \le n}\left\{\frac{i}{n} - F(x_i, \theta)\right\},\ D_n^- = \max_{1 \le i \le n}\left\{F(x_i, \theta) - \frac{i-1}{n}\right\},$$

n is the sample size, $x_1, x_2, \ldots, x_n$ are sample values in increasing order is usually used. The distribution of statistic (1) in testing simple hypotheses obeys the Kolmogorov distribution law $K(S)$.

In ω^2 Cramer-Mises-Smirnov test, one uses a statistic of the form

$$S_\omega = n\omega_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left\{F(x_i, \theta) - \frac{2i-1}{2n}\right\}^2, \qquad (2)$$

and in test of Ω^2 Anderson-Darling type, the statistic of the form

$$S_\Omega = -n - 2\sum_{i=1}^{n}\left\{\frac{2i-1}{2n}\ln F(x_i, \theta) + \left(1 - \frac{2i-1}{2n}\right)\ln(1 - F(x_i, \theta))\right\}. \qquad (3)$$

In testing a simple hypothesis, statistic (2) obeys the distribution $a1(S)$, and statistic (3) obeys the distribution $a2(S)$ [Bolshev and Smirnov, 1983].

In composite hypotheses testing, the conditional distribution law of the statistic $G(S|H_0)$ is affected by a number of factors: the form of the observed law $F(x, \theta)$ corresponding to the true hypothesis H_0; the type of the parameter estimated and the number of parameters to be estimated; sometimes, it is a specific value of the parameter (e.g., in the case of gamma-distribution and beta-distribution families); the method of parameter estimation. The distinctions in the limiting distributions of the same statistics in testing simple and composite hypotheses are so significant that we cannot neglect them. For example, Figure 1 shows distributions of the Kolmogorov statistic (1) while testing the composite hypotheses subject to different laws using maximum likelihood estimates (MLE) of two parameters, and Figure 2 represents a similar situation for statistic (3) distributions of the Anderson-Darling. Figure 3 illustrates statistic distribution dependence (2) of the Cramer-Mises-Smirnov test upon the type of parameter estimated by the example of Weibull law.

The paper [Kac *et al.*, 1955] was a pioneer in investigating statistic distributions of the nonparametric goodness-of-fit tests with composite hypotheses. Then, for the solution to this problem, various approaches where used [Durbin, 1976], [Martinov, 1978], [Pearson and Hartley, 1972], [Stephens, 1970], [Stephens, 1974], [Chandra *et al.*, 1981], [Tyurin, 1984], [Tyurin *et al.*, 1984].

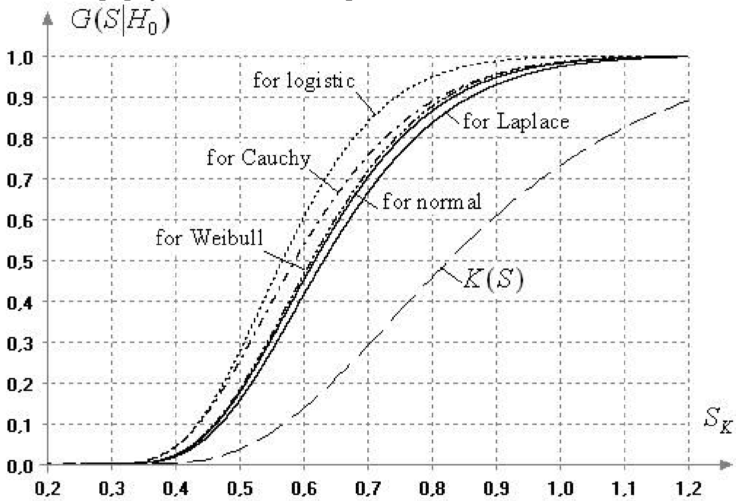

Fig. 1. The Kolmogorov statistic (1) distributions for testing composite hypotheses with calculating MLE of two law parameters

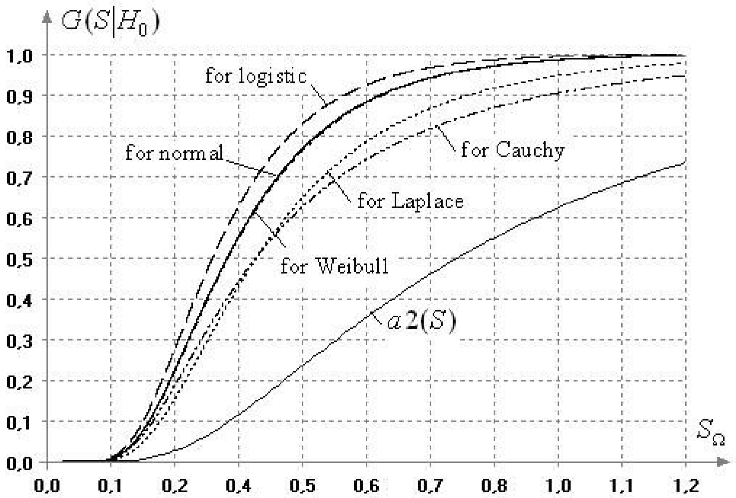

Fig. 2. The Anderson-Darling statistic (3) distributions for testing composite hypotheses with calculating MLE of two law parameters

In our research [Lemeshko, 1998], [Lemeshko and Postovalov, 2001], [Lemeshko and Maklakov, 2004], statistic distributions of the nonparametric goodness-of-fit tests are investigated by the methods of statistical modeling, and for constructed empirical distributions approximate models of law are found. The results obtained were used to develop recommendations for standardization [R 50.1.037-2002, 2002].

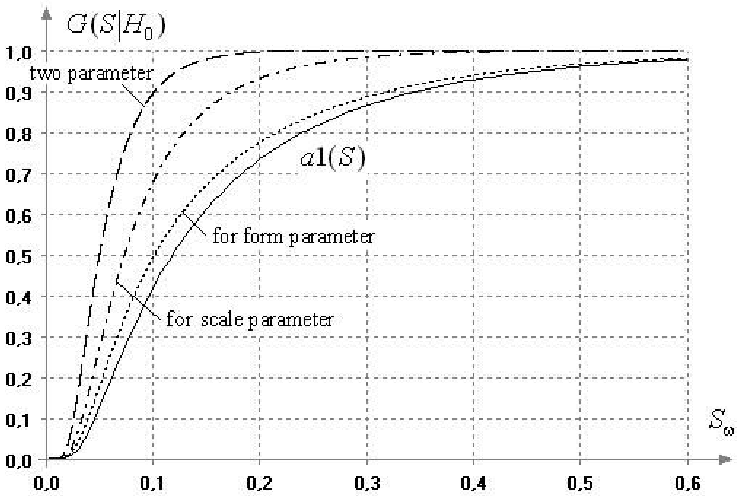

Fig. 3. The Cramer-Mises-Smirnov statistic (2) distributions for testing composite hypotheses with calculating MLE of Weibull distribution law parameters

In this paper we present more precise results (tables of percentage points and statistic distribution models) for the nonparametric goodness-of-fit tests in testing composite hypotheses using the maximum likelihood estimate (MLE). Table 1 contains a list of distributions relative to which we can test composite fit hypotheses using the constructed approximations of the limiting statistic distributions.

Upper percentage points are presented in Table 2, and constructed statistic distribution models are presented in Table 3.

Distributions $G(S|H_0)$ of the Kolmogorov statistic are best approximated by gamma-distributions family (see Table 3) with the density function

$$\gamma(\theta_0,\theta_1,\theta_2)=\frac{1}{\theta_1^{\theta_0}\Gamma(\theta_0)}\left(x-\theta_2\right)^{\theta_0-1}e^{-(x-\theta_2)/\theta_1}.$$

And distributions of the Cramer-Mises-Smirnov and the Anderson-Darling statistics are well approximated by the family of the *Sb*-Johnson distributions with the density function

$$Sb(\boldsymbol{\theta}) = \frac{\theta_1\theta_2}{(x-\theta_3)(\theta_2+\theta_3-x)}\exp\left\{-\frac{1}{2}\left[\theta_0-\theta_1\ln\frac{x-\theta_3}{\theta_2+\theta_3-x}\right]^2\right\}.$$

Random variable distribution	Density function $f(x,\theta)$	Random variable distribution	Density function $f(x,\theta)$
Exponential	$\frac{1}{\theta_0}e^{-x/\theta_0}$	Laplace	$\frac{1}{2\theta_0}e^{-\lvert x-\theta_1\rvert/\theta_0}$
Seminormal	$\frac{2}{\theta_0\sqrt{2\pi}}e^{-x^2/2\theta_0^2}$	Normal	$\frac{1}{\theta_0\sqrt{2\pi}}e^{-\frac{(x-\theta_1)^2}{2\theta_0^2}}$
Rayleigh	$\frac{x}{\theta_0^2}e^{-x^2/2\theta_0^2}$	Log-normal	$\frac{1}{x\theta_0\sqrt{2\pi}}e^{-(\ln x-\theta_1)^2/2\theta_0^2}$
Maxwell	$\frac{2x^2}{\theta_0^3\sqrt{2\pi}}e^{-x^2/2\theta_0^2}$	Cauchy	$\frac{\theta_0}{\pi[\theta_0^2+(x-\theta_1)^2]}$

Random variable distribution	Density function $f(x,\theta)$
Logistic	$\frac{\pi}{\theta_0\sqrt{3}}\exp\left\{-\frac{\pi(x-\theta_1)}{\theta_0\sqrt{3}}\right\}\Big/\left[1+\exp\left\{-\frac{\pi(x-\theta_1)}{\theta_0\sqrt{3}}\right\}\right]^2$
Extreme-value (maximum)	$\frac{1}{\theta_0}\exp\left\{-\frac{x-\theta_1}{\theta_0}-\exp\left(-\frac{x-\theta_1}{\theta_0}\right)\right\}$
Extreme-value (minimum)	$\frac{1}{\theta_0}\exp\left\{\frac{x-\theta_1}{\theta_0}-\exp\left(\frac{x-\theta_1}{\theta_0}\right)\right\}$
Weibull	$\frac{\theta_0 x^{\theta_0-1}}{\theta_1^{\theta_0}}\exp\left\{-\left(\frac{x}{\theta_1}\right)^{\theta_0}\right\}$

Table 1. Random variable distribution.

The tables of percentage points and statistic distributions models were constructed by modeled statistic samples with the size $N = 10^6$. In this case, the samples of pseudorandom variables, belonging to $F(x, \theta)$, were generated with the size $n = 10^3$.

Random variable distribution	Parameter estimated	Kolmogorov's test			Cramer-Mises-Smirnov's test			Anderson-Darling's test		
		0,1	0,05	0,01	0,1	0,05	0,01	0,1	0,05	0,01
Exponential and Rayleigh	Scale	0.9946	1.0936	1.2919	0.1743	0.2214	0.3369	1.0599	1.3193	1.9537
Seminormal	Scale	1,0514	1,1599	1,3811	0.2054	0.2659	0.4151	1.1882	1.4987	2.2666
Maxwell	Scale	0.9687	1.0615	1.2511	0.1620	0.2040	0.3063	1.0095	1.2474	1.8318
Laplace	Scale	1.1770	1.3125	1.5858	0.3230	0.4378	0.7189	1.7260	2.2861	3.6836
	Shift	0.9565	1.0444	1.2225	0.1513	0.1865	0.2671	1.0698	1.3005	1.8322
	Two parameters	0.8629	0.9398	1.0960	0.1152	0.1437	0.2136	0,7970	0,9818	1,4404
Normal and Log-normal	Scale	1.1908	1.3274	1.5999	0.3273	0.4425	0.7265	1.7450	2.3089	3.7061
	Shift	0.8881	0.9632	1.1136	0.1344	0.1654	0.2377	0.8923	1.0872	1.5510
	Two parameters	0.8352	0.9086	1.0566	0.1034	0.1257	0.1777	0.6293	0.7497	1.0297
Cauchy	Scale	1.1372	1.2748	1.5503	0.3155	0.4300	0.7113	1.7157	2.2769	3.6728
	Shift	0.9753	1.0700	1.2603	0.1722	0.2162	0.3185	1.2154	1.5121	2.2114
	Two parameters	0.8151	0.8926	1.0478	0.1287	0.1699	0.2708	0.9480	1.2260	1.9129
Logistic	Scale	1.1797	1.3158	1.5893	0.3232	0.4380	0.7190	1.7244	2.2845	3.6820
	Shift	0.8371	0.9072	1.0464	0.1191	0.1476	0.2163	0.8559	1.0427	1.4951
	Two parameters	0.7465	0.8054	0.9230	0.0813	0.0976	0.1352	0.5619	0.6645	0.9027
Extreme-value and Weibull	Scale [1)]	1.1823	1.3157	1.5832	0.3201	0.4311	0.7043	1.7234	2.2726	3.6343
	Shift [2)]	0.9949	1.0931	1.2922	0.1742	0.2212	0.3364	1.0591	1.3183	1.9522
	Two parameters	0.8243	0.8948	1.0366	0.1017	0.1235	0.1744	0.6336	0.7548	1.0396

Note. [1)] - we estimated the Weibull distribution form parameter, [2)] - the Weibull distribution scale parameter.

Table 2. Upper percentage points of statistic distribution of the nonparametric goodness-of-fit tests for the case of MLE usage.

Test	Random variable distribution	Estimation of scale parameter	Estimation of shift parameter	Estimation of two parameters
Kolmogorov's	Exponential and Rayleigh	γ(5.1092; 0.0861; 0.2950)	–	–
	Seminormal	γ(4.5462; 0.1001; 0.3100)	–	–
	Maxwell	γ (5.4566; 0.0794; 0.2870)	–	–
	Laplace	γ(3.3950; 0.1426; 0.3405)	γ(6.2887; 0.0718; 0.2650)	γ(6.2949; 0.0624; 0.2613)
	Normal and Log-normal	γ(3.5609; 0.1401; 0.3375)	γ(7.5304; 0.0580; 0.2400)	γ(6.4721; 0.0580; 0.2620)
	Cauchy	γ (3.0987; 0.1463; 0.3350)	γ(5.9860; 0.0780; 0.2528)	γ(5.3642; 0.0654; 0.2600)
	Logistic	γ(3.4954; 0.1411; 0.3325)	γ (7.6325; 0.0531; 0.2368)	γ (7.5402; 0.0451; 0.2422)
	Extreme-value and Weibull	γ(3.6805; 0.1355; 0.3350) [1)]	γ(5.2194; 0.0848; 0.2920) [2)]	γ(6.6012; 0.0563; .2598)
Cramer-Mises-Smirnov's	Exponential and Rayleigh	*Sb*(3.3738;1.2145; 1.0792; 0.011)	–	–
	Seminormal	*Sb*(3.527; 1.1515; 1.5527; 0.012)	–	–
	Maxwell	*Sb*(3.353; 1.220; 0.9786; 0.0118)	–	–
	Laplace	*Sb*(3.2262; 0.9416; 2.703; 0.015)	*Sb*(2.9669; 1.2534; 0.6936; 0.01)	*Sb*(3.768; 1.2865; 0.8336; 0.0113)
	Normal and Log- normal	*Sb*(3.153; 0.9448; 2.5477; 0.016)	*Sb*(3.243; 1.315; 0.6826; 0.0095)	*Sb*(4.3950; 1.4428; 0.9150; 0.009)
	Cauchy	*Sb*(3.1895; 0.9134; 2.690; 0.013)	*Sb*((2.359; 1.0732; 0.595; 0.0129)	*Sb*(3.4364; 1.0678; 1.000; 0.011)
	Logistic	*Sb*(3.264; 0.9581; 2.7046; 0.014)	*Sb*(4.0026; 1.2853; 1.00; 0.0122)	*Sb*(3.2137; 1.3612; 0.360; 0.0105)
	Extreme-value and Weibull	*Sb*(3.343; 0.9817; 2.753; 0.015) [1)]	*Sb*(3.498; 1.2236; 1.1632; 0.01) [2)]	*Sb*(3.3854; 1.4453; 0.4986; 0.007)
Anderson-Darling's	Exponential and Rayleigh	*Sb*(3.8386; 1.3429; 7.500; 0.090)	–	–
	Seminormal	*Sb*(4.2019; 1.2918; 11.500; 0.100)	–	–
	Maxwell	*Sb*(3.9591; 1.3296; 7.800; 0.1010)	–	–
	Laplace	*Sb*(4.3260; 1.0982; 27.000; 0.110)	*Sb*(3.1506; 1.3352; 4.9573; 0.096)	*Sb*(3,8071; 1,3531; 5,1809; 0,100)
	Normal and Log- normal	*Sb*(4.3271; 1.0895; 28.000; 0.120)	*Sb*(3.3085; 1.4043; 4.2537; 0.080)	*Sb*(3.5601; 1.4846; 3.0987; 0.080)
	Cauchy	*Sb*(3.7830; 1.0678; 18.0; 0.11)	*Sb*(3.4814; 1.2375; 7.810; 0.1)	*Sb*(3.290; 1.129; 5.837; 0.099)
	Logistic	*Sb*(3.516;1.054; 14.748; 0.117)	*Sb*(5.1316; 1.5681; 10.0; 0.065)	*Sb*(3.409; 1.434; 2.448; 0.095)
	Extreme-value and Weibull	*Sb*(3.512;1.064;14.496;0.125)[1)]	*Sb*(4.799; 1.402; 13.0; 0.085) [2)]	*Sb*(3.4830; 1.5138; 3.00; 0.07)

Note. [1)] - we estimated the Weibull distribution form parameter, [2)] - the Weibull distribution scale parameter.

Table 3. Models of limiting statistic distributions of nonparametric goodness-of-fit when MLE are used.

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