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Powers of Some Tests for Exponentiality

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Abstract

In the paper, some statistical tests for testing exponentiality have been considered. The distributions of tests statistics have been studied depending on sample sizes. Comparative analysis of the power of tests under different pairs of competing hypotheses has been conducted. Advantages and disadvantages of individual tests have been shown. Considered tests have been ranked by the test power.

Keywords: exponential distribution, hypothesis testing, test statistic, test power.

Introduction

The exponential distribution law is main distribution law used in reliability theory. Its analytical simplicity makes it attractive to engineers and researchers. However, you need to be sure, that behavior of observable random variable (for example, the moment's of product failure (breakdown)) is consistent by « desirable » exponential distribution before using this model. Otherwise, the benefit from computation simplicity will be repeatedly reduced by losses from conclusion incorrectness caused by deviation of empirical distribution from exponential distribution law.

There are a lot of papers devoted to exponential law, authors of these papers propose different statistical tests for testing hypothesis of exponentiality. The abundance of tests is caused by frequent use of exponential distribution model in applications. However, the frequency of using is defined that usage of simple model leads to the solution of problem grounded only on analytical methods in most cases.

In this paper, a lot of considered tests are studied by the method of statistical simulations. The number of experiments carried out for statistical modeling is assumed equal to 1 660 000 in the study of the distributions of test statistics.

1 The statement of exponentiality testing

Let $x_1, x_2, ..., x_n$ be sample of independent observations of nonnegative random variate X. Belonging of sample to exponential distribution law with density function $f(x) = exp(-x)$ was considered as tested hypothesis H_0.

The set of tests constructed special for exponentiality testing can be used for testing hypothesis H_0 besides classical goodness-of-fit tests. It is quite difficult to divide the special test statistics into the groups due to multiplicity of its. It should be noted that elements of ordered samples $x_1 < x_2 < ... < x_n$ are used in calculation for some test statistics. In another cases sequence order of elements doesn't matter.

Also, in some test statistics, we will use transformed values $z_1, z_2, ..., z_n$, which use estimate of shift and scale.

2 Alternative hypotheses

The exponential distribution has constant failure rate. The distribution laws corresponding to alternative hypotheses can be with: increasing, decreasing and non-monotonic failure rate [13]. The research was carried out for three alternative hypotheses:

$H_1 : LN(1)$ is lognormal distribution with denstity function $f(x) = \left(\theta x\sqrt{2\pi}\right)^{-1} exp\left(-(\ln x)^2/2\theta^2\right)$ and scale parameter $\theta = 1$ as alternative hypothesis with non-monotonic failure rate;

$H_2 : W(0.7)$ is Weibull distribution with density function $f(x) = \theta x^{\theta-1}exp(-x^{\theta})$ and form parameter $\theta = 0.7$ as alternative hypothesis with decreasing failure rate;

$H_3 : W(1.2)$ – is Weibull distribution with form parameter $\theta = 1.2$ as alternative hypothesis with increasing failure rate.

The distribution functions and density functions, which correspond to tested and alternative hypotheses, are presented on Figures 1 and 2, respectively.

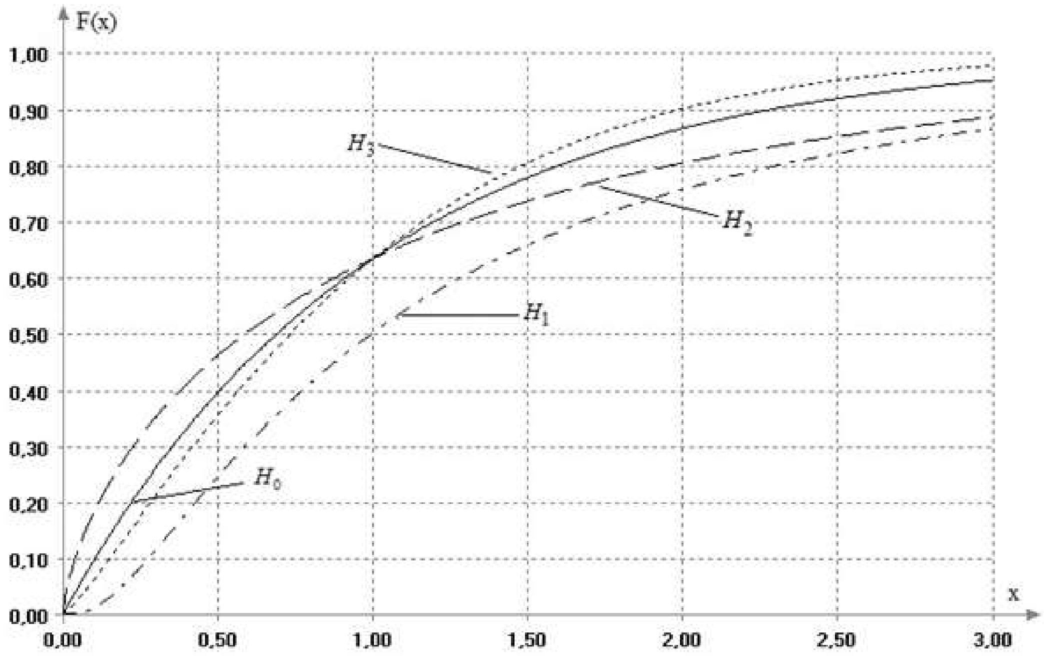

Figure 1: The distribution functions of hypotheses

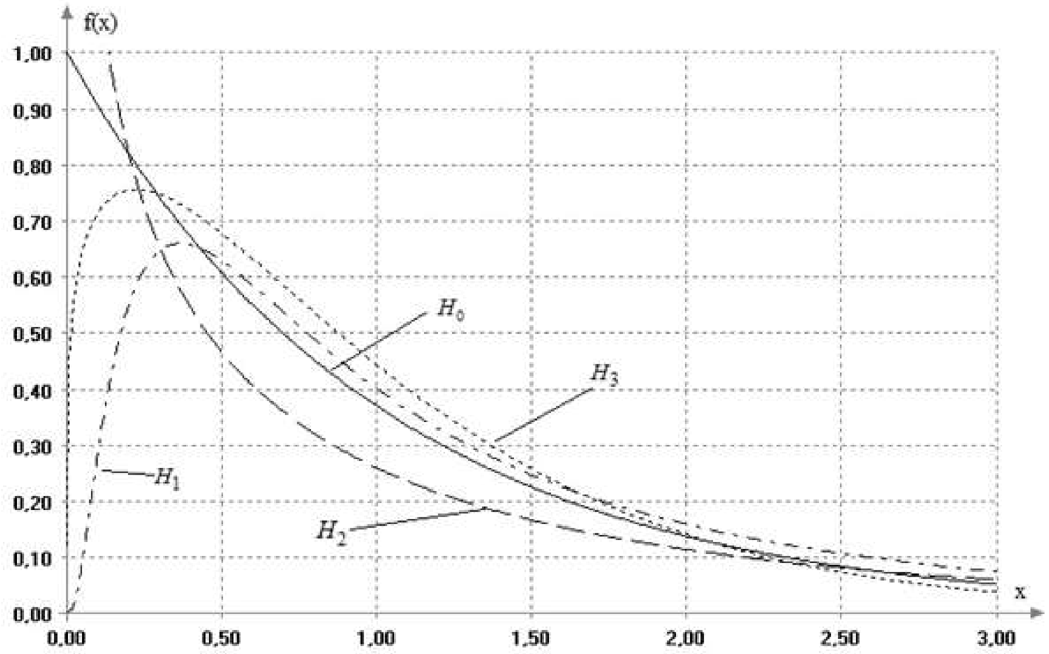

Figure 2: The density functions of hypotheses

3 Considered tests

In this section the description of considered tests are presented. The expressions for rest of test statistics of special uniformity tests are shown in Table 1, where $x_1 \leq x_2 \leq ... \leq x_n$ – ordered sample.

It is necessary to make the following remark. Although most of tests characterized as right-sided tests [8] by authors, but obtained results of research are contradicted to given assertions. The powers under one (or even two) considered alternative hypotheses with increasing of sample sizes leads to zero (values of statistic under validity of hypothesis H_i are less than under validity of hypothesis H_0) under supposition about right-sidedness of these tests. That fact can be described as follows: those tests considered like two-sided in power analysis. Only Frocini test, correlation tests and Kimber-Michael test are right-sided among analyzed tests, other considered tests are two-sided.

The correlation tests. Suppose that we have distribution law of probability $F(x) = 1 - exp\left(\frac{x-\mu}{\nu}\right)$ where μ and ν — non-known parameters, estimates of which can be calculate from formulas:

$$\hat{\nu} = \frac{n(\bar{x}-x_1)}{n-1};\ \hat{\mu} = x_1 - \frac{\hat{\nu}}{n}.$$

The test statistic based on correlation coefficient r for normalized variable $z_i = \frac{x_i-\hat{\mu}}{\hat{\nu}}$ with mathematical expectation i-th order statistic [14] of exponential distribution has the next form:

$$r(z,m) = \frac{\sum\limits_{i=1}^{n}(z_i-\bar{z})(m_i-\bar{m})}{\left[\sum\limits_{i=1}^{n}(z_i-\bar{z})^2(m_i-\bar{m})^2\right]^{1/2}},$$

where $m_i = \sum\limits_{j=1}^{i}\frac{1}{n-j+1}$. Under $n \geq 20$ approximation $\tilde{m_i} = -\ln\left(1-\frac{i}{n+1}\right)$ is possible and corresponding correlation coefficient is denoted by $r(z,\tilde{m})$.

The test statistics are used in form [8, 17]

$$K(z,m) = n\left[1-r^2(z,m)\right] \text{ and } K(z,\tilde{m}) = n\left[1-r^2(z,\tilde{m})\right].$$

Kimber-Michael test. Kimber has developed the test [6] based on linear dependence between theoretical $F(x)$ and empirical distribution function $F_n(x) = \frac{i}{n}$ of random variables. For standardized exponential law $F(z_i) = 1 - exp(-z_i)$, where $z_i = \frac{x_i}{\bar{\nu}}$, $\bar{\nu} = \frac{1}{n}\sum x_i$ is standardized random exponential variable, in order to stabilize dependence and reduce influence of non-equal dispersions, Michel proposed as follows [11]:

$$s_i = \arcsin\sqrt{F(z_i)},\ r_i = \arcsin\sqrt{\tfrac{i-0.5}{n}}.$$

The test statistic has form

$$D = \max_i |s_i - r_i|.$$

Hollander-Proshan test. The test statistic has form [5]

$$T = \sum_{i>j>k}\varphi(x_i, x_j + x_k),$$

$$\varphi(a,b) \quad = \quad \begin{cases} 1 & \text{if } a > b \\ 0 & \text{if } a < b \end{cases}$$

The normalized test statistic has form [5]:

$$T^* = \frac{T-M(T)}{\sqrt{D(T)}};\ \text{where } M(T) = \frac{n(n-1)(n-2)}{8},$$
$$D(T) = \frac{3n(n-1)(n-2)}{2}\left[\frac{5(n-3)(n-4)}{2592} + \frac{7(n-3)}{432} + \frac{1}{48}\right].$$

Klimko-Antle-Rademaker-Rockette test. The special exponentiality test based on equality testing of shape coefficient in Weibull distribution to one was considered in [7]. The test statistic can be written as follows

$$\tilde{c} = \sqrt{n}\left(\nu^{-1{,}075} - 1\right);\ \nu = \frac{\sqrt{\frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i-\bar{x})^2}}{\sum\limits_{i=1}^{n}\left\{x_i - \left[x_1 - \frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i-x_1)\right]\right\}}$$

where x_1 is first element of ordered samples or minimum.

Table 1: Statistics of considered tests

Number	Test	Test statistic
1	Shapiro-Wilk 1 [15]	$W_E = \frac{n(\bar{x}-x)^2}{(n-1)\sum\limits_{i=1}^{n}(x_i-\bar{x})^2}$
2	Shapiro-Wilk 2 [15]	$\widetilde{W}_{E_0} = \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{n\left[(n+1)\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right]}$
3	Frosini [3]	$B_n = \frac{1}{\sqrt{n}}\sum\limits_{i=1}^{n} \left\|1 - exp\left(-\frac{x_i}{\bar{x}}\right) - \frac{i-0.5}{n}\right\|$
4	Bartlett-Moran	$B = \frac{12n^2}{7n+1}\left[\ln\left(\frac{1}{n}\sum\limits_{i=1}^{n} x_i\right) + \frac{1}{n}\sum\limits_{i=1}^{n}\ln x_i\right]$
5	Sherman	$\omega_n = \frac{1}{2n}\frac{\sum\limits_{i=1}^{n}\|x_i-\bar{x}\|}{\bar{x}}$
6	Epps-Pulley	$c = \sqrt{48n}\left(\frac{1}{n}\sum\limits_{i=1}^{n} exp\left(-\frac{x_i}{\bar{x}}\right) - \frac{1}{2}\right)$
7	Max interval [1]	$\nu_n = \frac{\max(x_i - x_{i-1})}{\sum x_i}$, $i =$, $x_0 = 0$
8	Hartley [4]	$h(n) = \frac{\max x_i}{\min x_i}$
9	Kochar [9]	$T_n = \sqrt{\frac{108}{n}}\frac{\sum\limits_{i=1}^{n} J\left(\frac{i}{n+1}\right)}{\sum\limits_{i=1}^{n} x_i}$; $J\left(\frac{i}{n+1}\right) = 2\left(\frac{n+1-i}{n+1}\right)\left[1 - \ln\left(\frac{n+1-i}{n+1}\right)\right] - 1$

4 Simulation result

The Table 2 contains considered tests ordered by decreasing of power (quantity $1-\beta$) under alternatives H_1, H_2 and H_3 ($n = 50$ and $\alpha = 0.05$).

The best result are shown by Kimber-Michael test under hypothesis H_1, the power of this test are larger than powers of other tests considered. The correlation tests, Shapiro-Wilk tests, Frocini test and Klimko-Antle-Rademaker-Rockette demonstrate good powers as well.

It should be noted that worst results under hypothesis H_1 are shown by tests, limit distribution of which is normal distribution law (Epps-Pulley test [2], Sherman test [16], Hollander-Proshan test [5]).

The Bartlett-Moran test [12] shows the highest powers under alternative hypothesis H_2 and H_3. However, this test demonstrates average power under hypothesis H_1. The Epps-Pulley test, Frocini test and Hollander-Proshan test demonstrate consistently good ability to distinguish those alternative hypotheses from exponential distribution. The low power are presented by correlation tests (especially under hypothesis H_3), Hartley test and max interval test.

Table 2: The tests ranked by power ($n = 50$,$\alpha = 0.05$)

	hypothesis H_1	$1-\beta$	hypothesis H_2	$1-\beta$	hypothesis H_3	$1-\beta$
1	Kimber-Michael	0.516	Bartlett-Moran	0.890	Bartlett-Moran	0.315
2	Correlation test 2	0.377	Epps-Pulley	0.831	Epps-Pulley	0.304
3	Klimko-Antle-Rademaker-Rokette	0.359	Hollander-Proshan	0.818	Frosini	0.291
4	Shapiro-Wilk 1	0.359	Sherman	0.804	Hollander-Proshan	0.280
5	Correlation test 1	0.344	Frosini	0.804	Kimber-Michael	0.279
6	Frosini	0.310	Kochar	0.772	Sherman	0.277
7	Shapiro-Wilk 2	0.290	Kimber-Michael	0.706	Kochar	0.268
8	Max interval	0.254	Shapiro-Wilk 2	0.657	Shapiro-Wilk 2	0.266
9	Hartley	0.225	Klimko-Antle-Rademaker-Rokette	0.621	Klimko-Antle-Rademaker-Rokette	0.223
10	Kochar	0.218	Shapiro-Wilk 1	0.621	Shapiro-Wilk 1	0.223
11	Epps-Pulley	0.171	Max interval	0.347	Hartley	0.177
12	Bartlett-Moran	0.143	Hartley	0.319	Max interval	0.125
13	Sherman	0.140	Correlation test 2	0.311	Correlation test 1	0.053
14	Hollander-Proshan	0.109	Correlation test 1	0.276	Correlation test 2	0.016

Finally, it is significant that most powerful tests among considered tests are exponentiality Frocini and Kimber-Michael test.

Conclusions

Unfortunately, the distributions of most special uniformity tests depend on the sample size, therefore the possibility to estimate the significance level is absent and the researchers should use the tables of percent points.

In program system ISW, the possibility for simulation of the distributions of exponential test statistics in interactive mode is implemented as well as for simulation distribution of statistics of uniformity tests [10]. The estimate of p_{value} is calculated by empirical statistic distribution obtained as a result of modeling. This makes the statistical conclusions about the results of hypothesis testing more informative.

It is recommended to use some series of tests, which have certain advantages for more objective inferences.

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