A Review of the Properties of Tests for Uniformity

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Abstract **– A wide selection of tests for uniformity is considered. Distributions of test statistics under true null hypothesis are studied and power of tests is estimated by means of methods of statistical simulation. A comparative analysis of the power of tests is conducted. The conclusions are made on preference of one test or another under presence of specific competing alternatives.**

Index terms **– goodness-of-fit tests, uniformity test, uniform distribution, power of test, order statistics.**

I. Introduction

A NUMBER OF AUTHORS propose different statistical tests for testing a hypothesis of uniformity. The wide variety of tests is caused by frequent application of the uniform model in applications. This is not least defined by the fact that such a simple model makes it possible to solve problems based on analytical methods only.

Testing the uniformity actually represents goodness-of-fit testing the hypothesis of uniform distribution of the observed sample $x_1, ..., x_n$. Note that if $x_1, ..., x_n$ belong law with probability distribution function $F(x)$, then random variable $y_i = F(x_i)$ is uniformly distributed on the interval $[0,1]$.

Uniform distribution is often used to describe the measurement error of some instruments or measuring systems. Simulation of pseudorandom values according to different parametric laws relies on sensors of uniform pseudorandom values. Parametric laws are urgently needed in the systems of statistical simulation. All of these factors explain the increasing interest in the choice of simple and computationally efficient tests for hypotheses about the uniform law of analyzed samples.

In practice, the presence of a number of different tests states a complicated problem of the choice, i.e. information available in the literature does not definitely allow choosing a specific test.

In this paper, a lot of considered test studied by the method of statistical simulations. The number of experiments carried out for statistical modeling is usually assumed equal to 1 660 000 in the study of the distributions of the test statistic. One the one hand, such number of experiments allows tracing the qualitative picture of test statistic distributions in depend on various factors. In the other hand, this number of experiments provides acceptable accuracy of the power estimates and unknown probabilities. Computer analysis methods provide an opportunity to identify the advantages and disadvantages of a test, to assess the size of sample when the difference between distributions of test statistics under true tested hypothesis and the corresponding asymptotic (limiting) distributions of statistics is practically negligible. Also, these methods provide an opportunity to compare the relative powers of the different tests under various alternative hypotheses, and to identify the most preferable test.

II. Problem Definition

Suppose that $Rav(0,1)$ is the uniform distribution on the interval [0,1] and $x_1, ..., x_n$ are given independent observations of random variables. The hypothesis tested is $H_0 : X \in Rav(0,1)$. The hypothesis is composite, if the domain of the uniform random variable is determined by the sample.

The order statistics of X ($x_{(1)} < x_{(2)} < ... < x_{(n)}$ are elements $x_{(i)}$ of variation series of the sample) are used in the tests, let us denote these variables by U_i. Most tests considered can be divided in two groups. Test statistics in the first group based on the use of differences between close order statistics

$$D_i = U_i - U_{i-1},$$

where $U_0 = 0$, $U_{n+1} = 1$, n is the size of the sample.

Test statistics in the second group use differences between order statistic and mathematical expectation of this order statistic (or their modification).

III. Theory

A. Sherman test

Sherman test [1,6,7] refers to the first group, in which the difference between elements of order statistics is used. The test statistic is:

$$\omega_n = \frac{1}{2}\sum_{i=1}^{n+1}\left|D_i - \frac{1}{n+1}\right|.$$

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The null hypothesis H_0 is rejected for large values of statistic ω_n. Under true null hypothesis and large n, the distribution of test statistic ω_n is described by normal distribution. The estimate of the mathematical expectation and the variance for the distribution of Sherman test statistic are obtained from the results of statistical modeling and are presented in Table I.

TABLE I
MATHEMATICAL EXPECTATION AND VARIANCE FOR DISTIRIBUTION OF SHERMAN TEST STATISTIC

n	$E[\omega_n]$	$D[\omega_n]$	n	$E[\omega_n]$	$D[\omega_n]$
4	0.3277	0.0111	14	0.3552	0.0039
5	0.3349	0.0094	15	0.3561	0.0036
6	0.3399	0.0081	16	0.3568	0.0034
7	0.3436	0.0072	17	0.3574	0.0032
8	0.3464	0.0064	18	0.3580	0.0031
9	0.3488	0.0058	19	0.3585	0.0029
10	0.3505	0.0053	20	0.3589	0.0028
11	0.3520	0.0048	50	0.3643	0.0012
12	0.3532	0.0045	100	0.3661	0.0006
13	0.3543	0.0041	200	0.3670	0.0003

For $n \geq 20$, one can use normalized statistic

$$\omega_n^* = \frac{\omega_n - E[\omega_n]}{\sqrt{D[\omega_n]}},$$

where

$$E[\omega_n] = \left(\frac{n}{n+1}\right)^{n+1},$$

$$D[\omega_n] = \frac{2n^2 + n(n-1)^{n+2}}{(n+2)(n+1)^{n+2}} - \left(\frac{n}{n+1}\right)^{2n+2},$$

which is described by standard normal distribution as $n \to \infty$ [1,6,7]. These papers also provide another modification of the test statistic given by the formula

$$\tilde{\omega}_n = V - \frac{0,0955}{\sqrt{n}}\left(V^2 - 1\right),$$

where $$V = \frac{\omega_n - 0,3679\left(1 - \frac{1}{2n}\right)}{\frac{0,2431}{\sqrt{n}}\left(1 - \frac{0,605}{n}\right)}.$$

The distribution of this modified test statistic faster converges to the standard normal distribution than the distribution of original test statistic.

B. *Kimball test*

The Kimball test statistic [8] is similar to the test statistic of Sherman. The test statistic is

$$A = \sum_{i=1}^{n+1}\left(D_i - \frac{1}{n+1}\right)^2.$$

The hypothesis tested is rejected for large values of statistic A. This test roughly equivalent to Sherman test in the terms of power. The distributions of the statistic of Kimball test for different sample size n present in Fig. 1.

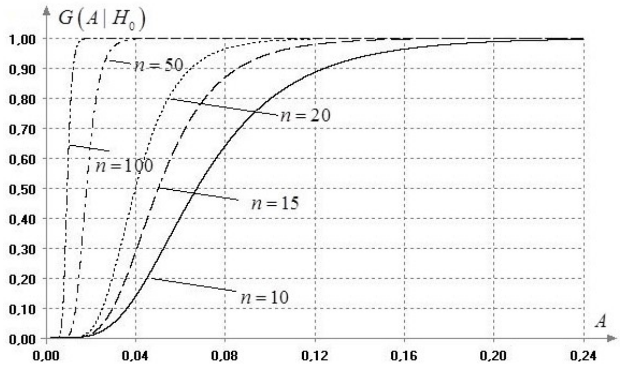

Fig. 1. Distributions of Kimball test statistic.

C. *Cheng and Spiring's test*

Cheng and Spiring's test statistic for uniformity [9] is calculated according to the following relation:

$$W_p = \frac{\left[\left(U_n - U_1\right)\frac{n+1}{n-1}\right]^2}{\sum_{i=1}^{n}\left(U_i - \bar{U}\right)},$$

where the difference $U_n - U_1$ is named as the range of sample.

If the H_0 is true, the following inequality always holds [1,9]:

$$\frac{2(n+1)^2}{(n-1)^2} \geq W_p \geq \frac{4(n+1)^2}{n(n-1)^2} \text{ (for even } n\text{)};$$

$$\frac{2(n+1)^2}{(n-1)^2} \geq W_p \geq \frac{4n(n+1)}{(n-1)^3} \text{ (for odd } n\text{)}.$$

Cheng and Spiring's two-sided test shows not high power as usually, however, the test shows the highest power in the case of the large sample size and alterna-tives, which are close to the uniform distribution.

D. *Hegazy and Green's test*

The type of the statistics used in these tests coincides with the type of statistics used in the corresponding tests for normality [10]:

$$T_1 = \frac{1}{n}\sum_{i=1}^{n}\left|U_i - \eta_i\right| \text{ и } T_2 = \frac{1}{n}\sum_{i=1}^{n}\left(U_i - \eta_i\right)^2 .$$

The differences between order statistics U_i and mathematical expectation $\eta_i = E[U_i]$ are used in statistics.

Tests can be used for random variables distributed on any interval $[a,b]$, but in this case it is necessary to replace U_i by $\frac{U_i - U_1}{U_n - U_1}$ and n by $n-2$ in formulas.

Thus, the test statistic has the form

$$T_1 = \frac{1}{m}\sum_{i=1}^{m}\left|y_i - \frac{i}{m+1}\right| \text{ и } T_2 = \frac{1}{m}\sum_{i=1}^{m}\left(y_i - \frac{i}{m+1}\right)^2 ,$$

where $m = \begin{cases} n, & when\ y_i = U_i \in [0,1]; \\ n-2, & when\ y_i = \frac{U_i - U_1}{U_n - U_1}. \end{cases}$

In [10], the modification of statistic is proposed, in the modification values $\xi_i = \frac{i-1}{m+1}$ are used instead of mathematical expectations $\eta_i = \frac{i}{m+1}$. This is due to the asymmetry of the distribution of order statistics. The modification test statistics is:

$$T_1^* = \frac{1}{m}\sum_{i=1}^{m}\left|y_i - \frac{i-1}{m+1}\right|;\ T_2^* = \frac{1}{m}\sum_{i=1}^{m}\left(y_i - \frac{i-1}{m+1}\right)^2 .$$

The hypothesis tested is not rejected with the significance level α, if inequalities $T_1 < T_1(\alpha), T_1^* < T_1^*(\alpha)$ or $T_2 < T_2(\alpha), T_2^* < T_2^*(\alpha)$ are respectively performed.

To find critical values of test, one can used the following approximation

$$T(\alpha) = a + \frac{b}{\sqrt{n}} + \frac{c}{n},$$

the values of coefficient a, b and c for $\alpha = 0{,}95$ and $\alpha = 0{,}99$ are shown in Table II. Critical values calculated by this formula coincide with critical values obtained by our simulation up to 2-3 decimal places.

TABLE II
VALUES OF COEFFICIENT FOR FINDING CRITICAL VALUES

Test statistics	Percentiles α					
	0,95			0,99		
	a	*b*	*c*	*a*	*b*	*c*
T_1	0,0003	0,5876	-0,0425	-0,0070	0,8373	-0,2500
T_1^*	0,0064	0,5066	0,2364	-0,0090	0,7949	-0,0782
T_2	-0,0068	0,0783	0,2419	-0,0148	0,1701	0,2745
T_2^*	0,0214	0,0214	0,8212	0,0047	-0,0607	0,9330

These tests show a good power.

E. Yang test

The test is designed to test the uniformity of samples distributed in the interval with length l.

The statistic of Yang test [11] is described by the formula

$$M = \frac{1}{l}\sum_{i=1}^{n}\min(D_i, D_{i+1}) .$$

For $l=1$ $M = \sum_{i=1}^{n}\min(D_i, D_{i+1})$. Since the value l is not always known, it is accepted to used the statistic in the form

$$M^* = \frac{1}{U_n - U_1}\sum_{i=2}^{n-1}\min(D_i, D_{i+1}) .$$

The distribution of this test statistic based on range of sample coincides with the distribution of test statistic M if n is replaced by $n-2$.

This test is two-sided. The hypothesis H_0 is rejected for both small and large values of the test statistic.

For $n \geq 15$ one can use the modification of test statistic

$$\tilde{M} = 2(n+1)M\sqrt{\frac{3}{2n-1}} - n\sqrt{\frac{3}{2n-1}} ,$$

which is belong to the standard normal distribution [11]. It was shown that this test has low power.

F. Greenwood, Quesenberry and Miller's test

The statistic of Greenwood test for uniformity is

$$G = (n+1)\sum_{i=1}^{n+1}\left(U_i - U_{i-1}\right)^2 .$$

Critical values of this test statistic equal to critical values of corresponding tests for exponentially, if n is replaced by $n-1$ [1,13].

The test of Greenwood, Quesenberry and Miller's test [12] with the statistic

$$Q = \sum_{i=1}^{n+1}\left(U_i - U_{i-1}\right)^2 + \sum_{i=1}^{n}\left(U_{i+1} - U_i\right)\left(U_i - U_{i-1}\right).$$

has a higher power [1].

The hypothesis under test is rejected for large values of this test statistic. The disadvantage of this test is the dependence of the distribution of the statistic on the size of samples n, if tested hypothesis H_0 is true.

G. Frozini test

The test statistic is:

$$B_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left|U_i - \frac{i-0.5}{n}\right|,$$

where U_i are order statistics based on sample $x_1, ..., x_n$. This test refers to second group. The hypothesis under test is rejected for large values of the statistic B_n. Critical values of the test statistic obtained by our simulation are presented in Table III. These values do not change for $n \geq 50$, it indicates the presence of the limit distribution.

TABLE III
CRITICAL VALUES OF FROCINI TEST STATISTIC

n	Percentiles α				
	0,8	0,85	0,9	0,95	0,99
3	0.3981	0.4345	0.4836	0.5596	0.6872
5	0.4027	0.4397	0.4895	0.5675	0.7161
7	0.4048	0.4423	0.4925	0.5726	0.7261
8	0.4052	0.4426	0.4931	0.5730	0.7292
9	0.4060	0.4433	0.4938	0.5737	0.7330
10	0.4064	0.4439	0.4943	0.5743	0.7326
12	0.4070	0.4442	0.4951	0.5759	0.7373
15	0.4075	0.4451	0.4958	0.5768	0.7398
17	0.4079	0.4453	0.4964	0.5780	0.7407
20	0.4083	0.4456	0.4966	0.5785	0.7428
50	0.4095	0.4472	0.4986	0.5808	0.7485
100	0.4100	0.4477	0.4991	0.5815	0.7506
200	0.4100	0.4477	0.4990	0.5816	0.7509

H. Neyman and Barton's test

This test is based on likelihood ratio [15]. The test statistic is based on values:

$$V_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \pi_j \left(U_i - 0.5\right),$$

where $\pi_j(y)$ are Legendre polynomials, which are orthogonal on the interval $[0,1]$. Generally, the first 4 polynomials are used:

$$\pi_1(y) = 2\sqrt{3}y; \quad \pi_2(y) = \sqrt{5}\left(6y^2 - 0,5\right);$$

$$\pi_3(y) = \sqrt{7}\left(20y^3 - 3y\right);$$

$$\pi_4(y) = 3\left(70y^4 - 15y^2 + 0,375\right).$$

The test statistic is

$$N_K = \sum_{j=1}^{K} V_j^2 .$$

The hypothesis under test is rejected for large values of statistics. In our study, test statistics with two, three and four polynomials of Legendre were used. Distributions of test statistics for $n = 200$ are present in Fig. 2.

In [16] it is shown, that the distribution of test statistic for $n > 20$ is approximated by $\chi^2 -$distribution with K degrees of freedom.

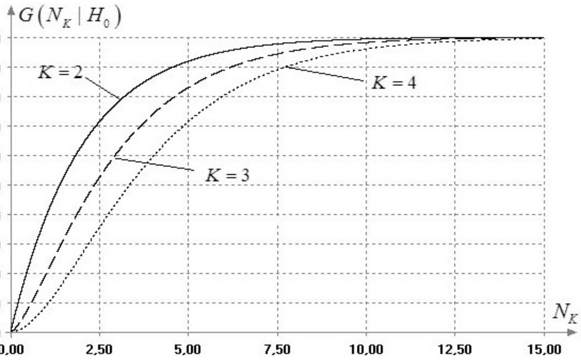

Fig. 2. Distributions of Neyman and Barton's test statistics N_K .

I. Kolmogorov and Smirnov's type test

The Kolmogorov and Smirnov's test with statistics D^+, D^-, D, and Kyper test V are used for testing the uniformity.

Test statistics of these tests are:

$$D^+ = \max_i \left(U_i - \frac{i}{n+1} \right), \quad D^- = \max_i \left(\frac{i}{n+1} - U_i \right),$$

$$D = \max\left(D^+, D^-\right), \quad V = D^+ + D^-.$$

Distributions of test statistics quickly converge to the limiting distributions, if we used statistics in the following forms:

$$\tilde{D}^+ = \left(D^+ + \frac{0,4}{n} \right) \left(\sqrt{n} + 0,2 + \frac{0,68}{\sqrt{n}} \right);$$

$$\tilde{D}^- = \left(D^- + \frac{0,4}{n} \right) \left(\sqrt{n} + 0,2 + \frac{0,68}{\sqrt{n}} \right);$$

$$\tilde{D} = \left(D + \frac{0,4}{n} \right) \left(\sqrt{n} + 0,2 + \frac{0,68}{\sqrt{n}} \right);$$

$$\tilde{V} = \left(V + \frac{1}{n+1} \right) \left(\sqrt{n+1} + 0,1555 + \frac{0,24}{\sqrt{n+1}} \right).$$

Distributions of the modification of test statistics for $n = 200$ are present in Fig. 3.

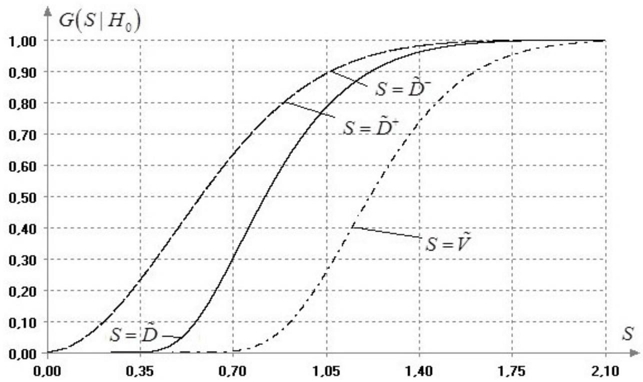

Fig. 3. Distributions of the modified test statistic.

IV. Experimental Results

Some authors give various normalizing transformations for test statistics, which make it possible to apply the standard normal law to normalized statistic to compute percentiles while testing the hypothesis. In practice, such asymptotical results may turn to be unacceptable for samples of finite volume as a consequence of significant difference between distribution of specific statistic and its asymptotical model.

We used the methodology of statistical simulation [16] to verify how close actual distributions of statistics fit to corresponding theoretical models. We investigated the distribution of 3-normalized statistics: two test statistics of Sherman and test statistic of Yang. In addition, test statistics of Neyman and Barton was checked for agreement with the chi-square distribution. The results are based on 16 600 simulations. The sample of test statistics obtained by our simulation was checked for agreement with their limiting distribution using 5 goodness-of-fit tests. Hypotheses of goodness-of-fit are not rejected for sample size $n \geq 20$. Normalized statistics of Sherman $\tilde{\omega}_n$ agrees with the standard normal law for $n \geq 10$.

V. Discussion of Results

We compared the power of tests for relatively sample size *n*=50, 100, 200. Empirical distributions of test statistics under either true null hypothesis or competing hypotheses were found based on 1 660 000 simulations. The hypothesis under test H_0 was chosen a uniform law. Alternative hypothesis H_1 was chosen beta distribution with the density

$$f(x) = \mathrm{B}(\theta_0, \theta_1)^{-1} x^{\theta_0 - 1} (1-x)^{\theta_1 - 1}$$

and the form parameters θ_0 and θ_1 are close to 1. This distribution was chosen because the fact that the standard uniform distribution is a special case of the beta distribution with the parameters of form θ_0 =1 and θ_1 =1.

The powers of tests obtained for sample size *n*=50, 100, 200 are given in Table IV-VI. Note that $\tilde{D}^{+}$ or $\tilde{D}^{-}$ demonstrate small values of power in some cases. This is due to the fact that the distribution functions of alternative hypothesis are located above or below the function of uniform distribution. (see Fig. 4).

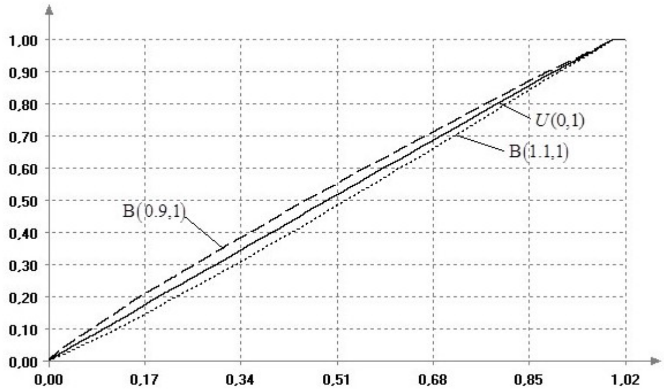

Fig. 4. – Distribution function of hypothesis H_0 and H_1

Table IV-VI also show the power in the case of $\mathrm{B}(1,2)$ and $\mathrm{B}(2,1)$ alternative hypothesis. These distributions are symmetric and far to uniform distribution. Problems of $\tilde{D}^{+}, \tilde{D}^{-}$ are more noticeable for these alternatives.

The power of tests obtained for alternative hypothesis, which is close to uniform distribution, is given in Table VII. Distribution functions of these alternatives crossed the function of the uniform distribution (see Fig. 5).

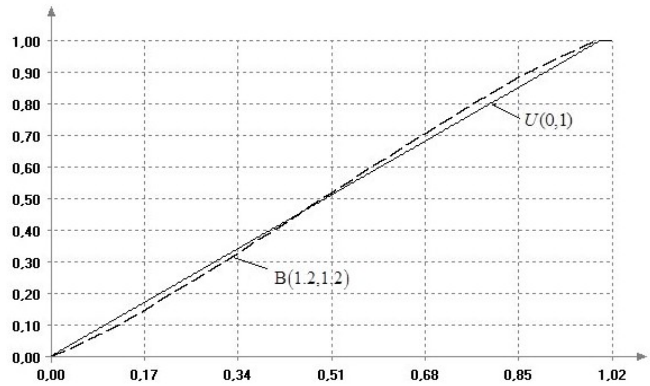

Fig. 5. – Distribution functions of hypotheses H_0 and H_1

Distributions $\mathrm{B}(1.05,1.05)$ and $\mathrm{B}(1.1,1.1)$ are close to the uniform distribution and therefore tests are not recognize the differences. Powers of test are obtained for smaller size of simulations 16 600 and for large sample sizes *n*=200 and *n*=1000. In this case, good power was shown by Cheng and Spiring's test and Neyman and Barton's test. This experiment was repeated for Cheng and Spiring's test on 1 660 000 simulations to clarify and confirm the results. Power of tests obtained on large size of simulation and power obtained on 16 600 simulation do match up to 3-4 decimal places.

VI. Conclusion

Studies have identified the strengths and weaknesses of considered tests. Obviously, among the all tests studied, we cannot unambiguously choose a test with the highest power with respect to every considered competing hypothesis. It is as well unrealistic to place the tests in some unconditional order, e.g., descending by power. In the same time, it is possible to select groups of tests useful in case of suggestion of certain kind of alternative.

Thus, with respect to competing distributions, which far from the uniform distribution, Hegazy and Green's tests and Neyman and Barton' tests show stably high power. The test with statistic based on the first two Lagrange polynomials is more powerful than other Neyman and Barton's tests. In some cases, the tests with modified statistic have the best power than other Hegazy and Green's tests. Tests with statistics T_1 and T_1^* show high power in situations when the distribution corresponding to the competing hypothesis is located above or below the function of the uniform distribution (see Fig. 4). Tests with statistics T_2 and T_2^* are more powerfull in situations when the distribution corresponding to the alternative hypothesis crossed the function of the uniform distribution (see Fig. 5),

Frosini test has the high power in comparison with Hegazy and Green's test and Neyman and Barton's test. The test with statistic $\tilde{D}$ and Kyper test are preferable among the Kolmogorov and Smirnov's type tests. Tests with statistics $\tilde{D}^+$ and $\tilde{D}^-$ can be used only in the case of certain kinds of alternatives.

The test of Sherman, the test of Kimball and the test of Greenwood, Quesenberry and Miller show less power than tests $\tilde{D}$ and $\tilde{V}$. Yang test has low power relative to any alternative hypotheses.

Generally, Cheng and Spiring's test demonstrate not high power. However, this the test of Kyper and Test of Neyman and Barton show the best power for alternative hypotheses, which close to uniform law, and large sample size.

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TABLE IV

POWER OF TESTS FOR UNIFORMITY WITH RESPECT TO COMPETING HYPOTHESES (n=50, α=0.05)

	$B(0.9,1)$	$B(0.95,1)$	$B(0.99,1)$	$B(1.05,1)$	$B(1.1,1)$	$B(1,2)$	$B(2,1)$
ω_n	0.0607	0.0539	0.0506	0.0483	0.0485	0.2652	0.2656
A	0.0602	0.0538	0.0506	0.0484	0.0487	0.3530	0.3533
W_p	0.0530	0.0511	0.0502	0.0497	0.0501	0.1112	0.1114
T_1	0.0738	0.0566	0.0506	0.0516	0.0606	0.7698	0.7704
T_1^*	0.0706	0.0553	0.0503	0.0527	0.0628	0.7864	0.7870
T_2	0.0727	0.0563	0.0507	0.0516	0.0601	0.7569	0.7572
T_2^*	0.0694	0.0551	0.0503	0.0526	0.0622	0.7790	0.7795
M	0.0520	0.0509	0.0502	0.0495	0.0494	0.0879	0.0875
$\tilde{D}^+$	0.0249	0.0358	0.0469	0.0677	0.0892	0	0.7988
$\tilde{D}^-$	0.1041	0.0727	0.0539	0.0339	0.0228	0.7992	0
$\tilde{D}$	0.0694	0.0555	0.0505	0.0511	0.0582	0.6802	0.6803
$\tilde{V}$	0.0642	0.0552	0.0507	0.0479	0.0482	0.3458	0.3465
B_n	0.0722	0.0559	0.0504	0.0521	0.0616	0.7801	0.7807
Q	0.0645	0.0555	0.0508	0.0473	0.0470	0.3831	0.3835
N_2	0.0748	0.0579	0.0509	0.0496	0.0547	0.6686	0.6683
N_3	0.0742	0.0577	0.0510	0.0485	0.0516	0.5595	0.5593
N_4	0.0740	0.0579	0.0511	0.0476	0.0495	0.4890	0.4888

TABLE V

POWER OF TESTS FOR UNIFORMITY WITH RESPECT TO COMPETING HYPOTHESES (n=100, α=0.05)

	$B(0.9,1)$	$B(0.95,1)$	$B(0.99,1)$	$B(1.05,1)$	$B(1.1,1)$	$B(1,2)$	$B(2,1)$
ω_n	0.0622	0.0539	0.0505	0.0493	0.0513	0.5292	0.5291
A	0.0628	0.0543	0.0505	0.0495	0.0523	0.7288	0.7288
W_p	0.0569	0.0522	0.0502	0.0506	0.0536	0.3586	0.3589
T_1	0.1032	0.0634	0.0508	0.0578	0.0842	0.9943	0.9942
T_1^*	0.0994	0.0620	0.0507	0.0591	0.0867	0.9952	0.9951
T_2	0.1010	0.0629	0.0509	0.0576	0.0827	0.9937	0.9936
T_2^*	0.0972	0.0615	0.0506	0.0588	0.0852	0.9949	0.9948
M	0.0517	0.0506	0.0501	0.0498	0.0500	0.1156	0.1157
$\tilde{D}^+$	0.0155	0.0289	0.0450	0.0814	0.1246	0	0.9941
$\tilde{D}^-$	0.1466	0.0875	0.0560	0.0275	0.0146	0.9943	0
$\tilde{D}$	0.0925	0.0608	0.0507	0.0560	0.0767	0.9833	0.9833
$\tilde{V}$	0.0754	0.0577	0.0509	0.0500	0.0564	0.8798	0.8799
B_n	0.1014	0.0628	0.0507	0.0584	0.0855	0.9948	0.9947
Q	0.0680	0.0560	0.0508	0.0487	0.0518	0.8154	0.8151
N_2	0.0998	0.0632	0.0511	0.0542	0.0728	0.9904	0.9904
N_3	0.0964	0.0625	0.0511	0.0521	0.0655	0.9790	0.9792
N_4	0.0948	0.0627	0.0514	0.0506	0.0609	0.9622	0.9623

Table VI

POWER OF TESTS FOR UNIFORMITY WITH RESPECT TO COMPETING HYPOTHESES (n=200, α=0.05)

	$B(0.9,1)$	$B(0.95,1)$	$B(0.99,1)$	$B(1.05,1)$	$B(1.1,1)$	$B(1,2)$	$B(2,1)$
ω_n	0.0680	0.0546	0.0504	0.0514	0.0580	0.9708	0.9709
A	0.0701	0.0553	0.0504	0.0522	0.0628	0.9992	0.9992
W_p	0.0776	0.0577	0.0506	0.0557	0.0759	0.9708	0.9709
T_1	0.2552	0.0980	0.0523	0.0884	0.2072	1	1
T_1^*	0.2501	0.0964	0.0520	0.0899	0.2112	1	1
T_2	0.2474	0.0959	0.0520	0.0868	0.2006	1	1
T_2^*	0.2416	0.0942	0.0519	0.0883	0.2049	1	1
M	0.0514	0.0504	0.0501	0.0502	0.0507	0.1574	0.1577
$\tilde{D}^+$	0.0038	0.0155	0.0403	0.1293	0.2678	0	1
$\tilde{D}^-$	0.3191	0.1409	0.0626	0.0149	0.0039	1	0
$\tilde{D}$	0.2152	0.0881	0.0517	0.0804	0.1743	1	1
$\tilde{V}$	0.1392	0.0704	0.0512	0.0512	0.0608	1	1
B_n	0.2528	0.0973	0.0521	0.0892	0.2093	1	1
Q	0.0787	0.0578	0.0506	0.0523	0.0655	1	1
N_2	0.2384	0.0917	0.0520	0.0785	0.1758	1	1
N_3	0.2214	0.0872	0.0520	0.0720	0.1504	1	1
N_4	0.2081	0.0843	0.0521	0.0671	0.1328	1	1

TABLE VII

POWER OF TESTS FOR UNIFORMITY WITH RESPECT TO COMPETING HYPOTHESES (n=200;1000, α=0.05)

$n=200$	$B(1.05,1.05)$	$B(1.1,1.1)$	$n=1000$	$B(1.05,1.05)$	$B(1.1,1.1)$
ω_n	0.049	0.052	ω_n	0.052	0.064
A	0.049	0.053	A	0.054	0.072
W_p	0,072	0,138	W_p	0.191	0.568
T_1	0.047	0.048	T_1	0.056	0.103
T_1^*	0.048	0.050	T_1^*	0.060	0.115
T_2	0.050	0.054	T_2	0.060	0.106
T_2^*	0.050	0.056	T_2^*	0.062	0.115
M	0.050	0.056	M	0.051	0.052
$\tilde{D}^+$	0.050	0.056	$\tilde{D}^+$	0.063	0.116
$\tilde{D}^-$	0.050	0.055	$\tilde{D}^-$	0.063	0.113
$\tilde{D}$	0.057	0.090	$\tilde{D}$	0.062	0.117
$\tilde{V}$	0,050	0,050	$\tilde{V}$	0.110	0.347
B_n	0.048	0.051	B_n	0.057	0.109
Q	0.048	0.051	Q	0.056	0.084
N_2	0.063	0.116	N_2	0.148	0.480
N_3	0.057	0.095	N_3	0.123	0.413
N_4	0.052	0.087	N_4	0.120	0.415